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REFLEXIONS

ON THE

METAPHYSICAL PRINCIPLES

OF THE

INFINITESIMAL ANALYSIS.

BY M. CARNOT.

TRANSLATED BY THE REV W. R. BROWELL, M.A.

FELLOW OF PEMBROKE COLLEGE, OXFORD.

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Although the treatise will be found particularly adapted for the Mathematical student, yet it is brought forward not without a hope that many, who are curious to know the principles of the New Calculus, without wishing to enter deeply into the science of abstract quantity, may find in it an answer to their enquiries. But few examples have been given out of the numerous list produced by the Author: they may all be found in the several elementary works on the Differential and Integral Calculus, and would, if introduced here, swell the volume unnecessarily.

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INFINITESIMAL CALCULUS.

IT is my object to ascertain in what the true spirit of the Infinitesimal Analysis consists. My reflexions on this subject will be divided into three chapters: in the first, I intend to point out the general principles of this Analysis: in the second, I shall examine how it has been reduced into Algorithm by the invention of the Differential and Integral Calculus: in the third, I shall compare it with the other methods, such as the method of Exhaustion, that of Indivisibles, of Indeterminate quantities, &c. which might be substituted for it.

CHAP. I.

General principles of the Infinitesimal Analysis.

1. THERE is no discovery which has produced in the science of Mathematics a change so important and so sudden, as that of the Infinitesimal Analysis: no method has furnished means more simple, or more successful, in promoting our knowledge of the laws of nature. By decomposing bodies (as it may be called) into their elements, it appears to have shewn their inward structure and organization: but as all extremes escape the senses and the imagination, none but an imperfect idea can be formed of these elements; a peculiar species of quantity, which at one time bear the character of real quantities, at another must be treated as absolutely nothing, and which seem by their equivocal properties to hold a middle rank between magnitude and zero, between existence and non-existence[a].

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[a] I am here speaking in conformity to the vague ideas that are commonly formed of quantities called Infinitesimals, when no pains have been taken to ascertain their nature: but, in reality, nothing is more simple than the exact idea of this sort of quantities.

Fortunately this difficulty has not impeded the progress of the discovery: there are certain primitive ideas which always leave some obscurity in the mind, but which, when their first deductions have once been made, open a field both extensive, and free from obstacles. Such we have seen to be the idea of infinity; and many geometricians, who never perhaps had investigated it, have made the most fortunate application of that idea: philosophers, however, could not content themselves with a notion so vague; they have been desirous of returning to principles, but they have found themselves divided in their opinions, or rather in their manner of regarding the objects of their speculation. My intention in this work is, to concentrate these different points of view, to shew their mutual relations, and to propose some afresh.

2. The difficulty which is often experienced in expressing exactly the different conditions of a problem by equations, and in solving these equations, might have given rise to the first idea of the Infinitesimal Calculus: indeed, when it is impossible to find the exact solution of a question, it is natural to endeavour to approach to it, as nearly as possible, by neglecting quantities which embarrass the combinations, if it be foreseen that

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these quantities which have been put aside cannot, by reason of their small value, produce more than a trifling error in the result of the calculation. For example, as the properties of curves are with difficulty discovered, it is necessary to consider them as polygons of a great number of sides. By way of illustration, if a regular polygon be supposed to be inscribed in a circle, it is evident, that these two figures, although always different, and incapable of becoming identical, are nevertheless more and more alike, according as the number of the sides of the polygon increases: it is evident, that their perimeters, their areas, the solids formed by their revolving round a given axis, the analogous lines drawn within or without these figures, the angles formed by these lines, &c. are, if not respectively equal, at any rate so much the nearer approaching to equality as the number of sides becomes increased; whence it follows, that, by supposing the number of these sides really very great, it will be possible, without any perceptible error, to assign to the circumscribed circle the properties that shall have been found belonging to the inscribed polygon. Besides, each of the sides of this polygon evidently diminishes in size, according as the number of the sides increases; and, consequently, if it be supposed that the polygon be really composed of a great number of sides, we may assert also, that each of them is really very small.

This being established, if by chance in the course of a calculation a particular circumstance were discovered, by means of which the operations might be greatly simplified, by neglecting (for example) one of these little sides in comparison with a given line, such as the radius; i. e. by employing in the calculation this given line instead of a quantity equal to the sum of the given line, and the small side spoken of above; it clearly might be done without impropriety; for the amount of error which would result from it could only be extremely small, and would not repay any pains bestowed on learning its value.

3. For example, let it be proposed to draw a tangent to a given point M, in the circumference MBD. (*Fig.* 1.) Let C be the centre of the circle, DCB the axis. Let us suppose DP $=x$, MP $=y$, and let TP be the subtangent sought.

In order to determine it, let us take the circle as a polygon of a great number of sides: let MN be one of these sides, let us produce it to the axis: this will clearly be the tangent in question,

[illegible]

[illegible]

If then x0 and y0 were known, we should know the value of y. Now these quantities x0, y0, are very small, since they are each less than one which by hypothesis is very small. Thus these quantities may be neglected in comparison with the quantities [illegible] and [illegible], to which they are added: the equation is then reduced to [illegible], as it ought to stand.

4. If this result be not absolutely exact, it is at any rate evident that in practice it may pass for such, since x0 and y0 are extremely small:

but any one who may have no idea of the doctrine of infinite quantities, will perhaps be surprised if he be told, that the equation $\mathrm{TP} = \frac{y^2}{a-x}$ not only approaches very near to the truth, but is really perfectly correct: it is however a fact, of which it is easy to be assured, by finding TP on the principle that the tangent is perpendicular to the extremity of the radius: for it is evident that the similar triangles CPM, MPT give

$$\mathrm{CP} : \mathrm{MP} :: \mathrm{MP} : \mathrm{TP}, \text{ or } \mathrm{TP} = \frac{\mathrm{MP}^2}{\mathrm{CP}} = \frac{y^2}{a-x}.$$

5. For a second example, let us suppose that it is proposed to find the area of a given circle. Let us again suppose this curve to be a regular polygon of a great number of sides: the area of any regular polygon whatsoever is equal to the product of its perimeter into the half of the perpendicular drawn from the centre upon one of its sides; hence the circle being considered as a polygon of a great number of sides, its area ought to equal the product of the circumference into half the radius; a proposition no less true than the result found above.

6. However vague then and wanting in precision the two expressions, Very great, and Very small, or others equivalent to them, may appear,

it is seen, by the two preceding examples, that they are not uselessly employed in mathematical combinations, and that their application may be highly advantageous in facilitating the solution of the various questions which may be proposed: for, once admit the idea of them, and all curves as well as the circle may be considered as polygons of a great many sides, all surfaces may be divided into a multitude of bands or zones, all solids into corpuscles; in a word, all quantities may be decomposed into particles of the same kind as themselves. Hence arise many new ratios and combinations; and we may easily judge, from the examples cited above, of the resources with which the introduction of these elementary quantities must furnish calculation.

7. But the advantage which they secure is much more considerable than as yet we have seen reason to expect: for it follows from the examples given, that what in the first place had been regarded but as a simple method of approximation, leads at any rate in certain cases to results perfectly exact. It would consequently be important to know how to distinguish those cases in which that happens, to reduce the others as much as possible to the former, and thus to change this method of approximation

into a calculus perfectly exact and rigorous. Now this is the object of the Infinitesimal Analysis.

8. Let us therefore now see how in the equation $\text{TP} = \frac{y\,(2y + \text{NO})}{2a - 2x - \text{MO}}$ found in (3) it has happened, that in neglecting MO and NO, the truth of the result has not been affected; or rather how this result has become exact by the suppression of these quantities, and why it was not so before. We may very easily assign a reason for what has occurred in the solution of the problem treated of above, by remarking, that the hypothesis from whence the reasoning started being false, (since it is absolutely impossible that a circle can ever be considered as a real polygon, whatever be the number of its sides,) there ought to have resulted from this hypothesis some error or other in the equation $\text{TP} = \frac{y\,(2y + \text{NO})}{2a - 2x - \text{MO}}$; and that the result $\text{TP} = \frac{y^2}{a - x}$ being no less certainly exact, as is proved by the comparison of the two triangles CPM, MPT, we may neglect MO and NO in the first equation, and we are even bound to do it in order to correct the calculation and to destroy the error which the

false hypothesis had introduced. To neglect quantities of this nature then is not only admissible in parallel cases, but it is necessary, and it is the only method of expressing exactly the conditions of the problem.

9. The exact result $\mathrm{TP} = \frac{y^2}{a-x}$ has then only been obtained by a compensation of errors: and this compensation may be rendered still more apparent, by treating the example given above in a somewhat different manner; that is to say, by considering the circle as a real curve, and not as a polygon. To do this, let a straight line RS parallel to MP be drawn from a point R, chosen arbitrarily at any distance from M, and through the points R and M let the secant RT^1 be drawn; we shall clearly have $\mathrm{T^1P} : \mathrm{MP} :: \mathrm{MZ} : \mathrm{RZ}$, and dividing $\mathrm{T^1P}$, $\mathrm{TP} + \mathrm{T^1T} = \mathrm{MP}\frac{\mathrm{MZ}}{\mathrm{RZ}}$.

Next, if we suppose RS to move *parallelly* to itself, by approaching continually to MP, it is evident that the point T^1 will approach, during the same time, more and more to the point T, and that consequently we may make the line TT^1 as small as we like, without disturbing the proportion established above. If then we neglect this quantity TT^1 in the equation just found, there will indeed result an error in the equation $\text{TP} = \text{MP}\,\frac{\text{MZ}}{\text{RZ}}$, to which the former will then be reduced: but this error may be diminished as much as we please, by causing RS to approach as near as required to MP; that is to say, the ratio of the two members of the equation will differ as little as we wish from the ratio of equality. In like manner we have

$$\frac{\text{MZ}}{\text{RZ}} = \frac{2y - \text{RZ}}{2a - 2x - \text{MZ}};$$

and this equation is perfectly exact, whatever be the position of the point R, i. e. whatever the values of MZ and RZ may be. But the more RS approaches MP, the smaller will MZ and RZ be: and dividing, if we neglect them in the second part of the equation, the error which will result from hence in the equation $\frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x}$, to which it will then be reduced, may like the first be made as small as shall be deemed necessary. This being the case, without regarding errors which may be always diminished as much as we please, we treat the two equations just found, viz.

$$\text{TP} = \text{MP}\,\frac{\text{MZ}}{\text{RZ}}, \text{ and } \frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x},$$

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In order to determine it, let us take the circle as a polygon of a great number of sides: let MN be one of these sides, let us produce it to the axis: this will clearly be the tangent in question, since this line will not extend into the interior of the polygon: let us also draw the perpendicular MO upon NQ parallel to MP, and call a the radius: thus we have MO : NO : : TP : MP, or $\frac{\mathrm{MO}}{\mathrm{NO}} = \frac{\mathrm{TP}}{y}$. Again, the equation of the curve being for the point M, $y^2 = 2ax - x^2$, it will be for the point N,

$$(y + \mathrm{NO})^2 = 2a\,(x + \mathrm{MO}) - (x + \mathrm{MO})^2.$$

Subtracting from this equation the former, which was found for the point M, and reducing it, we have

$$\frac{\mathrm{MO}}{\mathrm{NO}} = \frac{2y + \mathrm{NO}}{2a - 2x - \mathrm{MO}} = \frac{\mathrm{TP}}{y}$$

$$\therefore \quad \mathrm{TP} = \frac{y\,(2y + \mathrm{NO})}{2a - 2x - \mathrm{MO}}.$$

If then MO and NO were known, we should have the value of TP. Now these quantities MO, NO are very small, since they are each less than MN, which by hypothesis is very small. Then (2) these quantities may be neglected in comparison with the quantities $2y$ and $2x - 2a$, to which they are added: the equation is then reduced to $\mathrm{TP} = \frac{y^2}{a - x}$, as it ought to stand.

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$$\frac{\text{MZ}}{\text{RZ}} = \frac{2y - \text{RZ}}{2a-2x-\text{MZ}};$$

and this equation is perfectly exact, whatever be the position of the point R, i. e. whatever the values of MZ and RZ may be. But the more RS approaches MP, the smaller will MZ and RZ be: and dividing, if we neglect them in the second part of the equation, the error which will result from hence in the equation $\frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x}$, to which it will then be reduced, may like the first be made as small as shall be deemed necessary. This being the case, without regarding errors which may be always diminished as much as we please, we treat the two equations just found, viz.

$$\text{TP} = \text{MP}\,\frac{\text{MZ}}{\text{RZ}}, \text{ and } \frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x},$$

as if they were both of them perfectly exact;

by substituting in the latter the value of $\frac{\text{MZ}}{\text{RZ}}$ derived from the other, the result is as above,

$$\text{TP} = \frac{y^2}{a-x}.$$

This result is quite true, since it is in conformity with that which was obtained by the comparison of the triangles CPM, MPT; and yet the equations, $\text{TP} = y\frac{\text{MZ}}{\text{RZ}}$ and $\frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x}$ from whence it was deduced, are certainly both false, since the distance of RS from MP has never been supposed to vanish, nor even to be very small, but really equal to any line arbitrarily chosen. The errors consequently must compensate for one another, by the comparison of two faulty equations.

10. Here then is the fact of the compensation of errors ascertained and proved; it remains now that we explain it, and determine the signs by which we are assured that the compensation has taken place in calculations like the preceding, and the methods of producing it in each particular case. Now for this end it is sufficient to observe, that the errors made in the equations $\text{TP} = y\frac{\text{MZ}}{\text{RZ}}$ and $\frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x}$ admitting of being rendered as small as we please, that which would

occur, (if one were found in the resulting equation TP $= \frac{y^2}{a-x}$,) might with equal fairness be made as small as we please: and we may observe, that it would depend on the arbitrary distance of the lines MP, RS. But that is not the case, since the point M, through which the tangent must pass, being given, there are none of the quantities, a, x, y, or TP, which are arbitrary: hence it is impossible that there should be any error in this equation. A consequence of this is, that the compensation of the errors, occurring in the equations TP $= y \frac{\text{MZ}}{\text{RZ}}$ and $\frac{\text{MZ}}{\text{RZ}} = \frac{y}{a-x}$, shews itself in the result, by the absence of the quantities MZ, RZ, which occasioned these errors: and hence, that after having introduced these quantities into the calculation to facilitate the expression of the conditions of the problem, and after having regarded them as absolutely zero in comparison with the proposed quantities, with a view to simplify these equations, in order to banish the errors that they had occasioned, and to obtain a result perfectly exact, there remains but to eliminate these same quantities from the equations where they may still be.

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11. The inventor might then be led to his discovery by a very simple train of reasoning. If in the place of a proposed quantity, he might say, I employ in the calculation another quantity not equal to it, there will thence result some error or other: but if the difference of the quantities employed one instead of the other is arbitrary, and if I can make it as small as I please, this error will never be important: I might even make several similar errors at the same time without any bad result, so long as I should continue master of the degree of precision which I should wish to give to my results. Moreover, it might happen that these errors mutually compensate for one another, and that thus my results become perfectly exact. But how to effect this compensation in all cases? Some reflexion can alone discover this. Let us suppose, the inventor might say, that the compensation be made, and let us observe by what mark it ought to shew itself in the result of the calculation. Now what naturally must occur is, that the quantities which had occasioned the errors having disappeared, the errors also have disappeared; for the quantities such as MZ, RL having by hypothesis arbitrary values, they ought no longer to enter into any formulas or results which are not so, and which,

having become exact by supposition, depend alone not on the will of the calculator, but on the nature of those quantities, the ratio of which it was proposed to discover as expressed by these results. Hence the sign which announces that the desired compensation has taken place, is the absence of the arbitrary quantities which produced these errors, and hence to effect this operation, it remains but to eliminate these arbitrary quantities. Let us now attempt to give to these ideas the degree of precision and of generality which they merit.

DEFINITIONS.

12. Quantities are divided generally into Constant and Variable.

Quantities called Constant or Determined are those which have their values considered as fixed: and those which are called Variables or Indeterminate quantities are such as, on the contrary, admit of our assigning to them successively different values. But it must be remarked, that the expression of variable quantities is not to be taken in an absolute sense, since a quantity may be more or less indeterminate, more or less arbitrary. Now it is precisely on the different degrees of Indetermination of which quantity in general is susceptible, that all analysis, and more particularly this branch which is called the Infinitesimal Analysis, depends.

13. We will divide all quantities admitted into calculation into three classes. 1. Those that are Determined and Invariable by the very nature of the question. 2. Those which being at present Variable, afterwards take Determinate values by subsequent agreement or hypothesis. 3. Those which must always remain Indeterminate.

In the first of these three classes are what are called Constants or given quantities, such as the parameters of curves. In the second are the common Variables, such as the co-ordinates of curves, subtangents, normals, &c. to which we assign such or such Determinate values when we wish to discover their properties or ratios. In the third are those which being more or less independent of those in the two first classes, remain also more or less arbitrary until the operation is entirely completed, and which for this reason will be called quantities always Variable. But although quantities of this class remain always Variable, they are not on that account entirely arbitrary; and as simple Variables composing the second class are united to Constants of the first by any conditions or equations whatsoever, by virtue of which they cannot vary but according to certain laws; so "quantities always Variable" are united to common Variables and to given quantities as much by the conditions of the question as by the hypotheses on which the calculation rests, so that they themselves can only vary in a certain manner.

14. We will call every quantity, which is considered as continually decreasing, (so that it may be made as small as we please, without being at the same time obliged to make those quantities vary the ratio of which it is our object to determine,) an Infinitely small Quantity. When the ratio of certain proposed quantities is to be found, some constant, the others variable; we consider the whole system as having arrived at a determinate state regarded as fixed: in the next place we compare this fixed state with others of the same system, which are considered as continually approaching to the first until it differs from it as little as we please. These other states of the system are then, properly speaking, only auxiliary systems, which are interposed in order to facilitate the comparison between the parts of the former. The differences of quantities, which correspond in all these systems, may consequently be supposed as small as we please, without af-

fecting those which compose the former, and the ratio of which is the object of our enquiry. So that these differences have the nature of what we call Infinitely small Quantities: since they are considered as continually decreasing, and as admitting of being made as small as we please, without affecting the value of the sought unknown quantities. Unity divided by a quantity infinitely small is called an Infinite, or an Infinitely great Quantity.

Under the name of Infinitesimal Quantities are comprised Infinite *Quantities* as well as Infinitely small Quantities. The Infinitesimal Analysis is but the art of employing as auxiliaries Infinitesimal Quantities for the discovery of ratios between proposed quantities.

15. We now see then, by their character of mere auxiliaries, all the quantities called Infinitesimal, and any functions of them whatsoever, must necessarily be excluded from the results of the Calculus. For these results must be the expression of the ratios only prescribed by the conditions of the proposed question, and can only contain quantities between which ratios exist. So that the quantities which were only used as auxiliaries must no longer occur there. We introduced them at the beginning of the operation

merely to help the expression of the conditions; but that object once attained, they could not remain in the formulas, and must consequently be of necessity eliminated. It is besides their characteristic, that they can be employed only as auxiliaries; for their nature being always Variable, even when determinate values have been given to quantities the ratio of which the conclusion must express, if they continued in the result, they would cause it to vary when it ought to be fixed: and indeed the results of this new Analysis cannot be different from those of common Analysis: hence as this last never admits Infinitesimal Quantities, it is quite necessary that such as [illegible] be admitted into the former always conclude with being eliminated.

16. We observe from what precedes, that the quantities called Infinitely small in Mathematics are never quantities actually nothing, nor even quantities actually less than such or such determinate magnitudes; but merely quantities which the conditions of the proposed question, and the hypotheses on which the calculation is established, allow to remain variable until the operation is completed, by continually decreasing until they become as small as we wish, without being compelled at the same time to alter the values of

those, the ratio of which it is our object to obtain. It is in this alone that the true character of quantities which have received the name of Infinitely small Quantities consists, and not in the diminutiveness with which their name seems to suppose they are really endowed, nor in the absolute nullity which might be assigned them; and the idea, as we view it, is perfectly simple, and free from every vague and objectionable supposition.

17. In order to avoid circumlocution, we will henceforth include under the name of "Assigned Quantities," all those which compose the two first classes of which we have spoken, i. e. all those which are the subject of common analysis and the ratio of which we wish to obtain, or those which can possibly enter into the result of the operation. On the other hand, we will call "Unassigned Quantities" all such as compose the third class, i. e. those which remain always variable, and which are consequently more or less independent of those which compose the two first classes. Thus we must reckon Infinitesimal Quantities among Unassigned Quantities, and, after the definitions given above, it is easy to see, that quantities infinitely small, are but quantities "unassigned," which are supposed to

decrease gradually and simultaneously until they become as small as we please.

18. Let us apply all that has just been stated to the example already treated. The radius MC being given, (*Fig.* 1.) is a determinate quantity from the very commencement, by the nature of the question; thus it is " assigned," and in the first class of those we have mentioned (13). The lines DP, MP, TP, MT, are at present Variables, and become determinate only by the subsequent hypothesis, that the tangent must pass through the point M : but this supposition being once made, these quantities must be considered as fixed to the end of the calculation: thus they also are " Assigned Quantities," and belong to the second class of those above mentioned (13). These same quantities then, viz. the coordinates, the tangent and subtangent of the curve to the point M make up with MC, and those that are any functions whatever of them, the general system of assigned quantities, i. e. the sought quantities, and which can enter alone into the result of the calculation, or become the subject of ordinary algebra. On the other hand, the lines DQ, NQ, TQ, T^1Q, MZ, RZ, &c. are those which we have named " Unassigned Quantities," and which form the third class above mentioned (13), since

they always remain variable: for as we can always make MZ and RZ as small as we wish without changing the value of the " Assigned Quantities" spoken of above, these quantities MZ and RZ are such as we call Infinitely small, and the rest DQ, NQ, TQ, T^1P, TP, which are evidently functions of these infinitely small quantities, remain always variable in the same way, and are consequently such as we call " Unassigned Quantities."

19. Any two quantities whatsoever are said to differ in an infinitely small degree one from the other, when the quotient of one divided by the other differs from unity by only an infinitely small quantity.

We call a quantity " infinitely small with respect" to another quantity, when the quotient of the first divided by the second is an infinitely small quantity: and reciprocally the second is then called infinite or infinitely great relatively to the first.

One sees from hence, that a quantity even infinitely small may be infinitely great with respect to another such quantity: and reciprocally, that a quantity even infinitely great may be infinitely small with respect to another such. For if, by way of example, we suppose x an infinitely small quantity, x^2 will be a quantity infinitely small with respect to the first, although that first be itself infinitely small, since the ratio of the second to the first is $x : 1$, which by hypothesis is infinitely small. So if x represent an infinitely great quantity, it will nevertheless be infinitely small compared with x^2, since the quotient of the latter divided by the former will be x, a quantity by hypothesis infinitely great.

20. This observation enables us to divide Infinitesimal Quantities into different orders. If x, for example, is taken to represent an infinitely small quantity of the first order, x^2 will be one of the second order, x^3 of the third order, and so on.

Similarly, if x be taken to represent an infinitely great quantity of the first order, x^2 will be one of the second order, x^3 of the third order, and so on.

Two quantities of any order whatever are said to be of the same order, when their ratio is a finite quantity.

Whenever any two quantities added together or subtracted one from the other be found, the one infinitely small relatively to the other, the latter will be called the principal quantity, and the former the accessory quantity. Thus, for example, in the sum $\mathrm{x} + x$, of the quantities spoken of above, x is the principal and x the accessory quantity: and in the sum $x + x^2$, x is the principal and x^2 the accessory quantity.

21. As it is important that the different ideas introduced already be rendered familiar, we will again enter into some details on this subject.

The object of all calculation is to find the ratios existing between certain proposed quantities; but the difficulty of finding the ratios directly often demands the assistance of some other quantities, which when introduced never make a part of the proposed system, but which by their union with the first may serve as *intermediaries* between them.

We commence then by expressing the ratios that they bear to one another; after which we eliminate from the Calculus those which entered merely as auxiliaries, with a view to arrive at the actual ratios which we wished to discover between the proposed quantities alone. When amongst these auxiliary quantities any are found of such a nature, that we can make them all at the same time as small as we please, without making the proposed quantities vary during that same time; this circumstance gives rise incidentally to very important simplifications, and such as are the real source of the branch of calculation called Infinitesimal Analysis: and this is merely the art of choosing such similar auxiliaries, as are best adapted to the several cases, to use them in the most advantageous manner to express the conditions of the various questions, and to effect eventually the elimination of these same quantities, so that there may remain in the formulas those quantities alone, the ratios of which we wish to know.

22. This being established, let us suppose any system of quantities, some constant others variable, and that it be required to find the ratios or relations which exist between them. To ground our argument, let us begin by considering the whole system in any determinate state which we shall suppose fixed. Let us examine the ratios between the several quantities of this fixed system: these quantities and those which depend on them are exclusively what we call "Assigned Quantities." Let us now consider the proposed system in any new state whatever, different from the first; and since each of the quantities composing it is only an auxiliary quantity, (this new state in the mean while being supposed merely to find more readily the ratios of the quantities which constitute the first,) let us call this new

state of the system, an auxiliary system. Let us suppose that this auxiliary system approaches continually to the fixed system, because that all the auxiliary quantities composing the second state, approach simultaneously to the " Assigned Quantities" which correspond to them in the fixed system, and that in such a manner, that we can suppose all their respective differences as small as we please at the same time. These differences will be what we have called Infinitely small Quantities. As the quantities of this second system are purely auxiliary, they cannot enter into the result of the calculation, since this result is but the expression of the relations which exist between those which constitute the first: whence it follows, that the Infinitely small Quantities of which we have just spoken, and all their functions, are necessarily excluded from the same result.

23. Now if in the course of the calculation we had met with a constant quantity, and one of these infinitely small quantities added together, and if, considering that the latter may be taken as small as we please, whilst the other does not change, we had overlooked it in order to simplify the operation, as unimportant compared with the former, what would have been done? The natural conclusion would be without doubt, that the error

thus occasioned might be made as small as we please by diminishing more and more the arbitrary value of the quantity thus overlooked. But for this to be done, it is necessary that this arbitrary value itself, or some one of its functions, enter the result of the calculation: otherwise it would have no influence over it, and it could not consequently assist in correcting it by its continually diminishing. If then it is not there at all, this is a proof that the error will have corrected itself: for if after the process of the calculation it still remained, it could be only infinitely small. Now it cannot be so, since no Infinitesimal remains in the result: hence the error made in the course of the calculation must have disappeared in some way or other, and the following propositions will prove it rigorously.

Fundamental Principle.

24. Two quantities which are not arbitrary, cannot differ from one another by any other than a quantity which is not arbitrary.

Proof. Since the two proposed quantities are never arbitrary, we change neither the one or the other: hence we cannot change their difference: hence this difference is never arbitrary.

Corollary 1.

25. Two quantities that are not arbitrary are exactly equal to one another, whenever their supposed difference may be supposed as small as we please.

Let P and Q be the two proposed quantities: we have just seen that their difference cannot be arbitrary: it cannot therefore be supposed as small as we wish; which is contrary to hypothesis. Hence this supposed difference does not exist. Consequently P and Q are exactly equal.

Corollary 2.

26. Two "Assigned Quantities" are exactly equal, if it be proved that their difference is an "Unassigned Quantity."

"Assigned Quantities" are not arbitrary: hence their difference cannot be arbitrary: their difference therefore is necessarily an "Assigned Quantity:" hence, in order to prove that this difference does not exist, and that consequently the quantities are equal, it is sufficient to prove, that if it existed it could only be an "Assigned Quantity."

Corollary 3.

27. Every value that we may make as near as we please to the real quantity which it represents, without being at the same time compelled to change either the one or the other, is of necessity rigorously correct.

Since there is no occasion to change either the proposed quantity or its value in order to render the latter as near an approximation to the former as we wish: i. e. since we may regard both of them as fixed, and consequently non-arbitrary, in order that they may differ as little as we please from one another: therefore they fall under the case of Corollary 2. and are of necessity exactly equal.

Corollary 4.

28. Every quantity that we may suppose as small as we please may be overlooked as if it were absolutely zero, in comparison with every other quantity which cannot like the former be supposed to be as small as we please; without the errors (possibly arising hence in the course of the calculation) affecting the result, so soon as all the arbitrary quantities shall have been eliminated from it.

Whilst neglecting as absolutely zero the quantities which may be taken as small as we please, (since they are added to or taken from others which do not admit of being taken as small as we please,) it is clear that the errors which can arise in the course of the calculation, may similarly be considered as small as we please; there will remain then in the result some arbitrary quantity: but this is opposed to the hypothesis, since all arbitrary quantities are supposed to be entirely eliminated.

Corollary 5.

29. Every quantity, the relation of which to another quantity may be supposed as small as we please, may be neglected as absolutely zero in comparison with this last, without the errors (to which that would give rise in the course of the calculation) affecting the result so soon as all arbitrary quantities have been eliminated from it.

This Corollary is but an extension of the former. In Cor. 4. it was supposed, that those quantities in comparison of which we might overlook the others, could not themselves be supposed as small as we please: but in Cor. 5. we suppose, that both the one and the other may be made as small as we please, but that the ratio of one to the other is also susceptible of being taken as small as we please: consequently, of what nature soever both of the quantities may be, we may neglect (in comparison with the others) those quantities, the ratios of which to these last we may suppose as small as we please. And the demonstration is the same as that for Cor. 4; for it is evident, that if some errors arose from suppressing these quantities, we might always diminish them as much as we please, either in the course of the calculation, or in the result. Not that that can take place as respects this last case, for then some arbitrary quantity would be introduced there; and this is contrary to hypothesis, it being provided, that all arbitrary quantities should be eliminated.

30. The preceding propositions include the whole theory of the Infinitesimal Analysis: for those same quantities, which (according to the hypothesis on which the calculation is founded) may be rendered as small as we please, whilst the other quantities of the system cannot, are called "Infinitely small Quantities," and may consequently be neglected in the course of the calculation, as we have seen above, without the result being affected by it.

Leibnitz, the first to give the rules of the Infinitesimal Calculus, established it on this principle, viz. "that we might take arbitrarily one for the other two finite quantities, which differ from one another only by an infinitely small quantity. This principle had the advantage of great simplicity, and of a very ready application. It was adopted as a kind of axiom, and it was sufficient to regard these Infinitely small Quantities as less than all those which may be conceived or apprehended by the imagination. Soon did this principle effect wonders in the hands of Leibnitz himself, of the two Bernoulles, of L'Hopital, and of the rest. It was not however unassailed by objections. Leibnitz was reproached, first, with using the expression of Infinitely small Quantities, without having previously defined it: secondly, with having left it in some degree of doubt, whether he regarded his Calculus as absolutely rigorous, or as a mere method of approximation.

This illustrious writer, and the celebrated men who had adopted his idea, were satisfied with shewing, by the solution of the most difficult problems, the fertility of the principle, the constant agreement of its results with those of ordinary Analysis, and the superiority which it imparted to new calculations. These numerous instances of success triumphantly proved, all that objections were merely specious; but these learned men never answered them directly, and the difficulty remained as it was. These are truths which strike the minds of the discerning, and yet their demonstration mocks the efforts of the most skilful for a long time after their invention.

" M. Leibnitz," says D'Alembert, " embarrassed by the objections which he perceived might be raised to quantities infinitely small, such as the Differential Calculus considers them, preferred reducing his Infinitesimals to " quantities not to be compared;" which would overthrow the certainty of geometrical calculation."

But if Leibnitz was deceived, it must have been only in doubting about the exactness of his own Analysis, if indeed so great a man really entertained these doubts: a thing by no means probable; and he might answer,

First, You ask me what Infinitesimal quantities mean? I declare to you that I never by that expression mean metaphysical and abstract existences, as this abridged name seems to imply; but real, arbitrary quantities, capable of becoming as small as I wish, without being compelled at the same time to make those quantities vary whose ratio it was my intention to discover.

Secondly, You ask me if my calculation is perfectly exact and rigorous? I reply in the affirmative, so soon as I have arrived at eliminating from it the Infinitesimal quantities spoken of above, and have reduced it so as to contain ordinary Algebraic quantities alone. Up to that point I regard my calculation as only a simple method of approximation. Those, who in order to reconcile the rigour of the calculation throughout the operation with the simplicity of my Algorithm, have proposed considering the infinitely small quantities as absolutely nothing, never avoid the elimination just spoken of; and without attacking the force of their metaphysical reasoning, I remark, that they gain no advantage over me with respect to the simplicity of the operations, which are always the same ; and that they meet with another difficulty, viz. that all the terms of their equations vanish together, so that they have nothing but zeros to calculate, and the indeterminate ratios of o to o to determine.

Are not my Infinitesimal quantities, such as I have now proposed them, i. e. considered as less than all imaginable magnitude, *of equal value?* Are pure zeros more easily conceived? If you regard my valueless quantities as chimerical, may they not, as well as these pure zeros, be compared with one another? Have you a clearer idea of an imaginary quantity, such as $a\sqrt{-1}$, than of a *valueless* quantity? And yet do you hesitate to say that the ratio of $a\sqrt{-1}$ to $b\sqrt{-1}$ is $\frac{a}{b}$? Are not the mathematical sciences replete with similar enigmas? And are not these enigmas the distinguishing characteristics of synthetical Analysis? and do they not even furnish the former with those valuable resources, in which the latter is deficient? If I ask you what is the meaning of an equation in which imaginary quantities enter, as in the case of the equation of the third degree which does not admit of solution, you answer, that this equation can only assist in discovering the true value of the unknown quantity, when by any transformations whatever we have effected the elimination of the imaginary quantities. I make the same reply for my valueless quantities: I employ them only as auxiliaries: I allow that my calculation is not rigorously exact, until I have eliminated all: until that time it is not complete, and does not admit of application. Is yours more so, before you have cleared it of all your zeros? Moreover, in my new way of looking at the question, i. e. by considering my auxiliary quantities, not as altogether infinitely small, but only as indefinitely small, I shelter my Analysis from all cavil, I make it into a method, not of approximation, but of compensation; i. e. a method which combines the readiness of a simple approximating calculation with the exactness of the most rigorous methods, and I prove that it is nothing more than the method of exhaustion reduced to an algorithm. I am aware that we might supply its place by the method of exhaustion itself, by that of limits, or even by common algebra alone: but it should be ascertained whether these other methods combine in the same degree as mine does simplicity with fertility. On this point I refer to illustrious geometricians, who propose many other methods in theory, but who in practice avail themselves of mine.

31. But if it be *an advantage* to put aside such useless subtleties as are more calculated to embarrass the advance of science, than to supply them with a better foundation; it is also important to establish solid and immediately applicable principles on which our operations may rest. For the first condition to be fulfilled in Mathematics is to be exact; the second is to be as clear and simple as possible. There are

some persons, for example, who think they have sufficiently established the principle of the Infinitesimal Analysis, when they have reasoned thus: It is evident, (they say,) and acknowledged on all hands, that the errors to which the Infinitesimal Analysis might give rise, admit of being taken as small as we please, supposing there exist any at all. It is besides evident, that every error which we can suppose as small as we please, is in fact none: for since we may suppose it as small as we please, we may consider it zero: then the results of the Infinitesimal Analysis are rigorously exact.

This reasoning, plausible as it is at first sight, is nevertheless imperfect: for it is faulty to say, that since we may take an error as small as we please, we can with that view take it as absolutely nothing. For example, (*Fig.* 1.) the equation $\frac{\mathrm{MZ}}{\mathrm{RZ}} = \frac{y}{a-x}$, found in art. 9. is an equation always false, although we may reduce its error as much as we like, by continually diminishing MZ, RZ: for in order that this error may entirely disappear, it would be necessary to reduce these quantities MZ, RZ, to zero: but then the equation would be reduced to $\frac{0}{0} = \frac{y}{a-x}$, one that you cannot absolutely call false, but which is devoid of meaning, since $\frac{0}{0}$ is an indeterminate quantity. We must then either commit an error, however small it may be supposed, or come to a formula that teaches us nothing: and such is just the difficulty in the Infinitesimal Analysis.

32. Other persons confine themselves to considering the quantities called infinitely small as "incomparables;" in the sense that a grain of sand, for example, is in magnitude "incomparable" with the globe; for then, say they, the errors are valueless, and ought consequently to be entirely neglected in the result of the Calculus. But the Infinitesimal Analysis, looked at in this point of view, is only a method of approximation: at the same time that we know it to be quite rigorous. This comparison of the grain of sand to the globe may notwithstanding be useful to facilitate the expression of the conditions of the problem, by pointing out what may be neglected. But in concluding equations, the error, even of a grain of sand, ought no longer to be found. It should have disappeared in the same way that it was introduced, not once only in the course of the calculation, but several

times in the opposite processes, so that a compensation is necessarily effected, which discovers itself in a certain manner in final equations by the elimination of all arbitrary quantities.

33. The exactness of the principles of Leibnitz's Infinitesimal Analysis has been (it is presumed) sufficiently demonstrated; but in order to make the application of it more easy, they should be presented in a somewhat different light. Every equation is called "imperfect," the perfect exactness of which is not proved, but the error of which we nevertheless know that we may suppose as small as we please: that is to say, such, that in order to make this equation perfectly exact, it is enough to substitute for the quantities in it, or for some of them only, other quantities which differ from it infinitely little.

After this definition, it is clear that we may cause imperfect equations to undergo different transformations, without depriving them of the character of imperfect equations: as, for example, we may transpose the terms from one side to the other: we may multiply or divide these two sides by equal quantities; may raise them to the same powers, or extract the same roots from them.

Moreover, we may, instead of any quantities whatsoever contained in them, substitute others differing in an infinitely small degree; we may neglect infinitely small quantities relatively to finite quantities, and in more general terms " accessary" *quantities*, as compared with principal quantities; without these equations ever losing their primitive character of imperfect equations, and which become exact eventually by a compensation of errors.

But it is worth while to remark, that these continued errors, instead of removing us farther and farther (as it apparently must) from our object, (which is to reduce these imperfect equations to perfect exactness,) tend on the contrary to lead us to it by the shortest and simplest method, since by thus getting rid of these inconvenient " accessary" quantities in succession only, taking care not to affect the truth of the proposed equations, we at least disengage them from every consideration of infinity by the entire elimination of all arbitrary quantities, and observing that these quantities alone, the ratio of which we wished to know, be suffered to remain. This being laid down, the whole theory of infinite quantities may be comprised in the following theorem.

Theorem.

34. For an equation to be necessarily and rigorously exact, it is enough to know,

1. That it has been deduced from true or "imperfect" equations at the least, by transformations which have not deprived them of the character of imperfect "equations."

2. That it no longer retain any Infinitesimal quantity, that is to say, any other quantity than those of which it was proposed to find the ratio.

Demonstration.

Since by hypothesis the transformations to which we may subject the original equations have not deprived them of the character of equations, or of "imperfect" ones at the least, these equations cannot be affected by any other than errors admitting of being made as small as we please. But, on the other hand, these equations can no longer be such as have been called "imperfect;" for these can only exist between quantities which contain something arbitrary: since by their definition the error may be taken as small as we please. Now by hypothesis all arbitrary quantities are eliminated, since those alone remain of which we proposed to find the ratio.

The new equations then cannot be false, i. e. involve in them errors not admitting of being made as small as we please, nor such as have been called imperfect. Hence they are necessarily and rigorously exact.

Corollary 1.

35. Whether the proposed equations be expressed in Algebraic symbols, or whether instead of that they be propositions expressed in common language, the preceding proof always holds good. If then to arrive at the solution of any question whatever we establish our reasoning on such propositions as have resulting errors which admit of being considered as small as we please, and if at length, step by step, we arrive at propositions disengaged from all considerations of infinity, and consequently from all arbitrary quantities, these last propositions will be necessarily exact.

Corollary 2.

It follows from the Theorem and from the preceding Corollary, that the Infinitesimal Analysis may be reduced to three points, which, if strictly observed, can never lead to any other than quite exact results, and that by the most simple means known: viz.

1. To express the conditions of the proposed problem, either by exact equations, or at any rate by "imperfect" equations, or by equivalent propositions.

2. To transform these equations or propositions in different ways, without making them lose their primitive character of, at any rate, imperfect equations.

3. To direct these transformations with a view to completely eliminate the Infinitesimal quantities, and any functions whatever of these quantities, so as to free the results of the calculation from them entirely.

37. At the conclusion of this sketch of the doctrine of compensations, we may derive support for it in the opinion which that great man, (whose recent loss the scientific world is now mourning,) namely, Lagrange, entertained. The following are his expressions on this subject, in the last edition of his "Theorie des Fonctions Analytiques."

"It appears to me, that as in the Differential Calculus such as we use, we consider and calculate quantities actually infinitely small or supposed to be so, the true metaphysical principles of this calculation consist in this, that the error resulting from this false supposition is corrected or compensated for by that which springs from the very processes of the Calculus, according to which we retain in differentiation infinitely small quantities of the same order alone. For example, by regarding a curve as a polygon, with an infinite number of sides each infinitely small, and which when produced is the tangent of the curve, it is clear that we make a false supposition; but the error is corrected in the Calculus by our omission of infinitely small quantities. This we may easily observe in examples, but it would be perhaps difficult to give a general demonstration of it." This is a summary of all our theory, stated with more clearness and precision than we could command. Whether it be difficult or no to give a general proof of it, the true metaphysical principles of the Infinitesimal Analysis, such as we employ, and such as all geometricians agree must be employed in order to facilitate their calculations, are nevertheless, according to the illustrious author just quoted, the principles of the compensation of errors: and we are of opinion besides, that neither exactness nor general applicability are wanting in the demonstration of them which we have supplied.

38. The foregoing remarks contain merely the general principles of the Infinitesimal Analysis, and we will now apply them to some particular examples, before we illustrate the way in which these general principles were reduced into algorithm by Leibnitz, which has given them the character of a regular Calculus. Thus, what we have already advanced refers only to common synthesis and analysis, but this new synthesis is already in itself very important and clear: and if the ancients had possessed it instead of the method of exhaustion, which it superseded, they would have greatly simplified their labours, and would probably have pushed their discoveries much farther than they did: for they wasted their strength on overcoming difficulties, which were all at once surmounted by the mere idea of infinity.

As regards the use which the common Analysis of Algebra can make of the same idea, setting aside the algorithm which is peculiar to it, if it be desired to ascertain what power can be derived from it, it is only necessary to read Euler's " Introduction à l'Analyse des Infinis," and the power of a weapon like this in skilful hands will be found surprising.

Problem 1.

39. To draw a tangent to the common cycloid.

Let (*Fig.* 2.) AEB be a common cycloid generated by the circle E$pq$F. The chief property of this cycloid is that for any point m, the part mp of the ordinate contained between the curve and the generating circle, is equal to Ep, an arc of this circumference.

Next let us draw from this same circumference pT a tangent to the point p, and let our object be to find the point T, where this tangent will meet that of the cycloid mT.

Let a new ordinate nq be drawn infinitely near to the first mp, and let mr parallel to the small arc pq be drawn through m, which arc as well as mn will be considered as a straight line.

It is evident then, that the two triangles mnr, Tmp will be similar, and that consequently we shall have $mr : nr :: \mathrm{T}p : mp$. But since by the property of the cycloid, $\mathrm{E}q = nq$ and $\mathrm{E}p = mp$ $\therefore$ $\mathrm{E}q - \mathrm{E}p = nq - mp$ or $pq = nr$ or $mr = nr$. Hence in consequence of the proportion above, TP $=$ *mp* or TP $=$ E*p*, that is to say, that the subtangent T*p* is always equal to the corresponding arc E*p*. Now as this proposition is entirely free from all consideration of infinity, it is of necessity perfectly true.

Problem 2.

40. The volume of the paraboloid is half the cylinder of the same base and height.

Let (*Fig.* 3.) A*mn*C be the generating parabola, TA*p*D its axis, A*p* the absciss answering to the point *m*, *pm* the ordinate, T*p* the subtangent, *qn* a second ordinate infinitely near the first, A*rs* the tangent to the vertex, *mr*, *ns* two perpendiculars drawn from the points *m* and *n* on this tangent. Let *mt* be *mr* produced until it meets with *qn*.

The curve is considered as a polygon of an infinite number of sides, and *mn* as one of them. By imagining the figure to turn round the axis TA*p*, the small trapezium *pmnq* will produce one of the elements of the paraboloid, and the small trapezium *rsmn* the corresponding element of the volume, which makes up the complement of the paraboloid, with reference to the cylinder generated by the quadrilateral figure $\mathrm{A}snq$. These two elements are equal. For it is clear that the first, that is to say, that of the paraboloid, (if we neglect the quantities which are infinitely small with reference to the rest,) is $qp \times pm \times \frac{1}{2}$ circ. pm, and that the other is $rs \times mr \times$ circ. $\mathrm{A}r$. But $rs = tn$, $\mathrm{A}r = pm$, and since in the parabola the subtangent is double the absciss $mr = \frac{1}{2}\,p\mathrm{T}$. The second element then expressed above, becomes $tn \times \frac{1}{2}\,p\mathrm{T} \times$ circ. pm.

Now the similar triangles mnt, $\mathrm{T}mp$ give $tn : mt :: pm : \mathrm{T}p$; and $tn \times \mathrm{T}p = mt \times pm$: by substituting this value of $tn \times \mathrm{T}p$ in the preceding expression, it will become $mt \times pm \times \frac{1}{2}$ circ. pm, or $pq \times pm \times \frac{1}{2}$ circ. pm, which is the same as that found above for the first element. These two elements then are equal, or differ from one another in an infinitely small degree.

But as this equality remains still affected by infinite quantities, we will suppose the whole paraboloid to be composed of similar elements, and applying to each the same reasoning as heretofore, the conclusion is, that the sum of all the elements of the paraboloid, i. e. the volume of this body, is equal to the sum of the elements of the complemental volume, and consequently but half of the cylinder: a proposition which as it is quite free from all consideration of infinity, is of necessity entirely true.

CHAP. II.

On the Algorithm adapted to the Infinitesimal Analysis.

41. The general principles of the new doctrine being once well laid down, we may remark in the numerous applications of which it is capable, that amongst the infinitely small quantities employed, there is amongst them one particular class, which offer themselves to our notice more frequently than all the rest: these are what are called "Differentials."

By the word "differential" we mean the difference of two successive values of the same variable, when we consider the system to which it belongs in two or more states following, one of which is supposed fixed, and the rest as approaching continually and simultaneously to the first, until they differ from it in as small a degree as we please.

42. The expression, "a differential quantity," as it is a diminutive, points out at once that it expresses a difference, and that this difference is an infinitely small quantity. It marks the in-

finitely small quantity by which the variable value has increased, by passing from its first to its second state.

The differential of a quantity is generally expressed in the Calculus by the letter *d* put before that which expresses the variable: thus dx means the differential of x; dy the differential of y; $d\,\frac{x}{y}$ the differential of the fraction $\frac{x}{y}$, that is to say, the infinitely small quantity by which this fraction is increased, when x is increased by dx, and y by dy. The letter *d* does not then represent a quantity, but is employed as a mark or sign: it is only an abbreviation of the "differential of," and bears in the Calculus the name of the "Characteristic."

Constant quantities have no differentials, or rather their differentials are zero; since, by reason of their nature, they do not increase, or their increase may be supposed nothing, when the system is considered as passing from one state to another.

43. When the Calculus gives as the differential of a quantity a negative value, it is a proof that we have made a false supposition, and that the variable, instead of increasing, as we have supposed it to do, does, on the contrary, decrease,

by the general change in the state of the system. Thus, for example, an arc of the circle less than the fourth of the circumference being represented by *s*, its differential is *ds*, that of its sine will be *d* sin. *s*, and of its cosine *d* cos. *s*. Now, as we suppose *s* to go on increasing, it is clear that the sine will also increase, but that the cosine will diminish. Hence the Algebraical Calculus ought to give a negative value to *d* cos. *s*, as it does indeed.

But whether the variable goes on increasing or diminishing, we always understand by the differential, the difference between its second and first state, and we always mark it by the characteristic *d* followed by the variable, and taken positively; leaving it in common to the Calculus itself to correct the false suppositions which we may have made. When several variables are united by any law whatsoever, as, for example, the absciss and ordinate of a curve are, the increase of the one necessarily determines the increase of the other. Thus calling *x* the absciss and *y* the ordinate, there is between *dx* and *dy* a relation determined by that of *x* and *y* themselves. And, reciprocally, the relation of *x* and *y* depends on that which they have to their differentials *dx* and *dy*.

Hence the two branches of the Infinitesimal Analysis, the one having for its object to find the ratio between differentials of several variables and these variables themselves, when we know that which exists between the latter only: the other to retrace back the ratio which exists between the variables alone, when we are acquainted with that which unites these variables with their differentials.

44. It is easily imagined how the rules of these calculations, being once discovered, may aid in solving the several problems that can be proposed. For every problem amounts to nothing more than finding the ratio which exists between certain " Assigned" quantities. Now if it be difficult to detect at once this ratio, it is natural to endeavour to arrive at it by the intervention of some auxiliary quantities: but of all auxiliary quantities, experience teaches us that none provide us with more simplifications than those which are called Infinitesimal; it is then natural to introduce them as much as possible in combinations. Then it turns out that either they eliminate themselves, like common algebraic quantities, and in this case all the processes gone through in the course of the operations belong to what is called the " Dif-

ferential Calculus;" or it will be necessary to have recourse to certain transformations not used in common Algebra, but the object of which always is to eliminate these auxiliaries called Infinitesimals; and these transformations come under the direction of what is called the "Integral Calculus."

The first of these is much more easy than the second; since it includes, properly speaking, no process which is not common to the old analysis. But the Integral Calculus requires very different processes, and such as are far from being complete, notwithstanding the labours of scientific men of the first order, who have employed themselves in them. The object of this work is only to illustrate the spirit of these methods, and to point out the general course of these calculations. We will take the differential first, as the simplest and as indispensable in arriving at a knowledge of the integral Calculus: but we will restrict ourselves in both one and the other to primary ideas.

On the Differential Calculus.

45. We have said, that the differential of a variable quantity was the infinitely small dif-

ference between the second and first state of that variable: it remains then to find this differential for all possible cases; i. e. for all possible functions of the proposed variables, such as x, y, z, &c. the differentials of which have been already expressed by dx, dy, dz, &c.

We must examine what distinction we ought to make between the operation by which we should take a common or finite difference, and that by which we must limit ourselves in taking a differential or an infinitely small difference. If we consider the proposed difference in any two determined states whatever, differing one from the other, the difference of the two values of the same quantity taken in the two systems will be determined, and cannot therefore be supposed as small as we please: thus we could not neglect it without making a fault which we should not be able to rectify. But if the two systems are supposed to approach one another as much as we please, the difference of the two values of the same variable may be also made as small as we please, and it will become what is called a differential, and will be nothing but the common difference simplified by the suppression of quantities which in their expression might be found to be infinitely small with respect to other terms of which it is composed. Such is the principle of Differentiation.

46. It evidently follows from this general principle, that in order to differentiate a quantity, or any function whatever of this quantity, or of several quantities in combination, expressed thus $\phi\ (x, y, z, \&c.)$, we have only to consider it in the second state, i. e. when x, y, z, &c. become respectively $x+dx, y+dy, z+dz$, &c. and this function becomes $\varphi\ (x+dx, y+dy, z+dz, \&c.)$; in the next place, to subtract from this function, thus increased, what it was before, i. e. $\varphi\ (x, y, z, \&c.)$ which will give no difference of the proposed function,

$$\varphi\ (x+dx, y+dy, z+dz, \&c.) - \varphi\ (x, y, z, \&c.)$$

and then in order to pass from this difference to the differential, we have only to reduce the expression by neglecting the quantities which are infinitely small, compared with those to which they are added or from which they are subtracted[*].

On the Integral Calculus.

47. We must not forget, that Infinitesimal quantities are never any other than auxiliary quantities

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[*] It is unnecessary to enter into the detail of differentiation, which may be examined in the numerous treatises on this subject.

introduced into the Calculus solely to facilitate the comparison of "assigned quantities," i. e. of those whose relation we wish to discover: and that the ultimate object is always to eliminate them. When this operation requires the common transformations of Algebra alone for its completion, the process refers to what is called the "Differential Calculus." But when we can obtain this elimination only by the converse operation of that which we went through in order to differentiate the proposed quantities, this operation becomes the subject of what is called the Integral Calculus.

To integrate a differential quantity, is to find that quantity again which by its differentiation gives the differential quantity proposed.

But this converse operation is much more difficult than the direct, in the same way that division is more difficult than multiplication, of which however it is only the converse operation: as the extracting roots is more complicated than raising to powers, of which this is also the converse; and, lastly, as solving equations is more difficult than making them, since we do not arrive at it even generally, except for the inferior degrees.

There are besides a great many differential expressions, such as ydx, which cannot really result from any differentiation, and which consequently cannot be integrated: there are others which may be susceptible of integration, but the means of effecting it are not yet known.

48. The quantity which by its differentiation produces a proposed differential, is called the integral of this differential; since we consider it as having been formed by infinitely small continual additions: each of these additions is what we have named the differential of the increasing quantity, it is a fraction of it: and the sum of all these fractions is the entire quantity which we are in search of, and which for this reason is called the " integral" of this differential: for the same reason we call " Integrating" or " taking the sum of" a differential, the finding the integral of the sum of all the infinitely small successive additions which form the series, the differential of which, properly speaking, is the general term.

49. An integral being considered as the sum of the elements called differentials, it has been agreed on to mark it in the Calculus by the characteristic ($\int$), an abbreviation of " the sum of." Thus the sign of ($\int$) destroys the effect of the sign (d,) so that $\int dx$ is nothing more than the quantity x itself.

It is evident, that two variables which constantly remain equal, increase the one as much as the other during the same time, and that consequently their differences are equal: and the same holds good even when these two quantities had differed by any quantity whatsoever, when they began to vary; provided that this primitive difference be always the same, their differentials will always be equal.

Reciprocally, it is clear, that two variables which receive at each instant infinitely small equal additions, must also remain constantly equal to one another, or always differ by the same quantity: i. e. that the integrals of two differentials which are equal can never differ from each other, but by a constant quantity. For the same reason, if any two quantities whatever differ in an infinitely small degree from each other, their differentials will also differ from one another infinitely little: and reciprocally, if two differential quantities differ infinitely little from one another, their integrals, putting aside the constant, can also differ but infinitely little one from the other.

50. On these principles is founded the application of the rules of the Integral Calculus. By way of example, let AMNR be a curve,

the area of which, that is to say, the surface AMP contained between the arc AM of this curve, its absciss AP, and its ordinate MP, we wish to find.

If we suppose the absciss AP or x to increase by the infinitely small quantity PQ or dx, the area sought will increase by the small trapezium MNQP: this small trapezium will then be the element or differential of the surface sought: thus we shall then have

$$\text{AMP} = \int \text{MNQP}. \qquad (\text{A})$$

But, on the other hand, by neglecting the small triangle MNO, which is evidently infinitely small with regard to the trapezium, we shall have as area of the trapezium, considered equal to the rectangle MOQP, the product of y its base into dx its height, or ydx: then ydx differs infinitely little from the differential of the area sought. Hence (49) $\int ydx$ will differ infinitely little from $\int$MNQP or AMP, without the constant, or

$$\text{AMP} = \int ydx + \text{C}, \qquad (\text{B})$$

which is what has been called an imperfect equation. Let us suppose that the curve is a common parabola, the parameter of which is p, we shall then have $y^2 = px$, and differentiating,

$2ydy = pdx$, or $dx = \frac{2ydy}{p}$: substituting this value in equation (A) we have

$$\text{AMP} = \int \frac{2y^2dy}{p} + \text{C}$$

$$\text{but } d\,\frac{2y^3}{3p} = \frac{2y^2dy}{p}$$

$$\therefore \text{AMP} = \frac{2y^3}{3p} + \text{C}$$

or since $y^2 = px$,

$$\text{AMP} = \tfrac{2}{3}\,xy + \text{C}.$$

But this equation, which so far has been regarded as imperfect, contains no longer an Infinitesimal quantity. Hence it has become perfectly rigorous; i. e. the area of the parabola is exactly equal to $\tfrac{2}{3}\,xy + \text{C}$.

It remains for us to determine C. To do this, the origin of the abscisses being in A, if we suppose $x = o$, we shall also have $y = o$. And as we are finding the whole area, measuring from A, we shall have also $\text{AMP} = o$. Hence the equation $\text{AMP} = \tfrac{2}{3}\,xy + \text{C}$, holding good for every value of x, gives

$$o = o + \text{C}, \text{ or } \text{C} = o,$$

and the equation which gives the area of the parabola, is reduced to $\text{AMP} = \tfrac{2}{3}\,xy$.

We may judge from this example the use

which we may make of the integral Calculus, and how important it is to trace back the means of passing from differential equations, which we may have found by the expression of the conditions of the problem, to integral equations, which can be derived from them.

51. The rules of the integral Calculus are necessarily derived from those of the differential, of which it is the converse. The detail of these rules cannot enter into a work like this.

On the Calculus of Variations.

52. The Calculus of Variations is one of the most brilliant conceptions of our immortal Lagrange. The principal object of this Calculus is to solve generally the well-known questions of Maxima and Minima, which for so long a time after the invention of the Infinitesimal Analysis occupied the first geometricians of Europe.

Euler had already treated these questions with the depth and clearness usual to him, in a separate work, called, "Methodus inveniendi lineas curvas maximâ minimâve proprietate gaudentes." But it is to Lagrange that we owe the Algorithm which has invested this beautiful theory with a peculiar character, and with a process as uniform and simple as possible.

In the common questions relative to Maxima and Minima, it is required to find the determinate values which we must assign to different variables which enter into such or such a finite proposed function of these same variables, so that this function may attain the greatest or least possible value.

In the Calculus of variations, on the contrary, the ratios between the variables are required, i. e. the equations which must obtain between these variables, in order to satisfy the condition of maximum or of minimum: besides the function which must be a maximum or minimum is not, as in common questions, solely composed of finite quantities, but it must be the integral simply marked of a differential function which is not to be integrated.

53. For example, let it be required to draw on a plane, (*Fig.* 4.) between two given points A, D, such a curve that AK being the axis, and DK the ordinate, drawn perpendicularly to it from the point D, the area contained between the curve and the coordinates AK, DK, may be a maximum amongst all the curves of the same length.

Although this problem belongs to the general theory of Maxima and Minima, we nevertheless see that it is of a very different nature from the ordinary questions; for it is not required here to find the determinate values AK DK of x and y, which satisfy the proposed conditions; since these values are already given; but the thing is to find the nature itself of the curve, i. e. the general equation which must hold good for every point between the coordinates.

The area contained within this curve, whatsoever it be, and the extreme coordinates AK, DK, is $\int ydx$: it is then this simply marked integral which must be a maximum.

Thus in this class of questions it is, as we stated above, the integral of a certain differential quantity not to be integrated, generally, which ought to be a maximum, and it is required to find the ratio which must exist between the variables to satisfy this condition.

53. However, the general principle is always the same as for ordinary questions of Maxima and Minima, that is to say, that when the quantity which must become a maximum has arrived at that term, it can no longer increase. Whilst it approaches this term it increases by degrees until it has attained it: then it becomes as it

were, stationary, previously to its beginning to diminish insensibly:* so that at this stage of maximum the quantity may be considered as constant, or having for its differential zero, although those of the variable quantities of which it is a function are not. Hence if the form of the curve varied infinitely little to become AM′N′D, and if we mark the new coordinates by x', y'; the area $\int y'dx'$ might be considered equal to $\int ydx$; or, which amounts to the same thing, the addition which $\int ydx$ acquires to become $\int y'dx'$ must be supposed equal to o, when the relation of x to y is that which corresponds with the maximum sought. Now this addition is what is called the variation of $\int ydx$.

54. Let then AM′N′R′D be any curve whatever, differing infinitely little from the first: if we consider this curve as the curve AMNRD itself, which undergoes an infinitely small transformation, we can suppose each point M′ of the transformed, as a point M of the other which has passed from M to M′, so that each of the points of the new curve has its correspondent point in the former. In the next place, the addition, which each of these quantities entering into the system receives in the change as it passes from its first to its second state, is called its "variation." Thus the varia-

tion of MP as M′P′ — MP, that of NQ is N′Q′ — NQ, that of the whole curve is

AM′N′R′D — AMNRD ;

that of the surface is

AM′N′R′DKA — AMNRDKA ;

and so with the rest.

55. Although the variation of a quantity is the difference of the two values of the same quantity differing infinitely little from one another, we must not confound it with its differential: for the latter is the difference of two consecutive values taken upon the same curve, whilst the variation is the difference of the two values taken on the two curves: thus the differential of the absciss is AQ — AP, whilst its variation is AP′ — AP: the differential of the ordinate is NQ — MP, whilst its variation is M′P′ — MP, &c.

In order not to confound these two classes of differences, we mark the new kind, i. e. the variation, by the characteristic δ, whilst the differential retains for its characteristic the letter *d*. Thus PQ $= dx$, PP′ $= \delta x$, NQ — MP $= dy$, and M′P′ — MP $= \delta y$, &c.

When we pass from the point M to N on the first curve, we pass from M′ to N′ on the second: thus, PQ being the differential of x, i. e. the quantity by which x increases, when we pass from M to N; P′Q′ will be the differential of AP′, i. e. the quantity by which AP′ or $x + \delta x$ increases at the same time. Likewise N′Q′—M′P′ is the differential of M′P′ or of $y + \delta y$.

56. Variations are only introduced, like differentials, as auxiliaries, to assist in discovering the ratio which really does exist between the coordinates of the first curve. It will then be necessary to eliminate all these variations after having introduced them, so that there may only remain the ratio between the coordinates sought for: or, if you please, first between these coordinates and their differentials, and then, by means of integration or otherwise, between the coordinates alone and the constants which compose the system of "assigned quantities."

57. We have already remarked, that the combinations of analysis in general are founded on the several degrees of indetermination, that the quantities assembled together in one calculation possess. We have a fresh example of it here. For variations are quantities more indeterminate than differentials, and these are more so than simple variables.

When we conceive the nature of a curve to change infinitely little, we always regard the first curve as a fixed term, to which we refer it in its different successive stages. The small changes effected are shewn by means of the variations that the coordinates and other quantities of the system undergo, and these variations may be supposed as small as we please without changing the "assigned system," whilst that which is given by variations is an auxiliary system, which approaches continually to the first, until it differs from it as little as we please. The differentials are subjected to the law proscribed to them by the ratio of the coordinates, whilst the law which connects the variations with these coordinates is arbitrary; whence it follows, that although the one and the other are auxiliaries only, the latter are more indeterminate than the former, since they may still change even when we regard the former as fixed.

58. If we suppose x to be one of the variables, and x' the quantity differing infinitely little from it, which corresponds to it in the new system, $(x'—x)$ will be the infinitely small quantity, by which x will have been increased in order to become x', and consequently the same as what we have named the variation of x, or δx. It is then evident, that in order to find the variation of any function whatever of x, we shall have merely to substitute $(x + \delta x)$ instead of x, then to subtract

the former function from it: a process which, being the same as that of differentiation, shews that there is no difference between variation and differentiation, except in the characteristic, which is δ for the former and d for the latter.

CHAP. III.

On the methods which may be substituted for the Infinitesimal Analysis.

59. There are several methods of solving questions, dependent on the Infinitesimal Analysis: and although there is not any which appears to combine the same advantages, it is nevertheless interesting to enquire, what are the different points of view from which this theory may be regarded: for this reason it is proposed to observe shortly upon the different methods which refer to it, and which may even be substituted for it.

On the method of Exhaustion.

60. The method of Exhaustion was employed by the ancients in their difficult investigations, and particularly in the theory of lines and curved surfaces, and in taking the value of areas, and the volumes of solid bodies contained by them. As they admitted perfectly rigorous demonstrations alone, they thought they could not allow them-

selves to consider curves as polygons of a great number of sides: but when they wished to discover the properties of one of them, they regarded it as the fixed boundary to which the inscribed and circumscribed polygons approach continually, and as much as they pleased, according as they increased the number of their sides. Thus they exhausted in some measure the space comprised between these polygons and the curve, which doubtlessly gave to this operation the name of "the method of Exhaustion." As these polygons terminated by straight lines were known figures, their continual approach to the curve gave an idea of it more and more precise, and the law of continuity serving as a guide, they could eventually arrive at the exact knowledge of its properties.

But it was not sufficient for geometricians to have observed, and as it were guessed, at these properties: it was necessary to verify them in an unexceptionable way; and this they did by proving, that every supposition contrary to the existence of these properties would necessarily lead to some contradiction: for this reason, this kind of demonstration was called "reductio ad absurdum."

61. It was thus that, having established as

true that the areas of similar polygons are to one another as the squares of their homologous lines, they hence concluded, that circles of different radii are to one another as the squares of these radii; which is the second proposition of the twelfth book of Euclid. Analogy conducted them to this conclusion, by imagining in these circles regular polygons to be inscribed of the same number of sides. For, as by increasing as much as they pleased the number of these sides, their areas remained always to each other as the squares of the radii of the circumscribed circle, they easily perceived that the same thing must necessarily obtain, for the circles themselves to which these polygons approach more and more until they differ from it as little as they pleased: but this was not sufficient; it was necessary to prove rigorously that the fact was really so: and this they did by demonstrating, that every contrary supposition necessarily led to an absurdity. The ancients proved in the same way that the volumes of spheres are to one another as the cubes of their diameters; that pyramids of the same height are as their bases; that the cone is one third of a cylinder of the same base and height.

The ancients, in order that they might the better effect their object, would often introduce as auxiliaries the inscribed and circumscribed polygons at once, which they compared together: thus they contracted more and more the curves comprised between these rectilinear figures, and seized more easily on the properties of this mean quantity.

62. They used it also with regard to curved surfaces and the volumes of bodies. They supposed them all at once inscribed and circumscribed by other surfaces, the number of the sides and zones of which they gradually increased, so as to inclose the proposed surface more and more between the two. The law of continuity pointed out to them besides the properties of this mean figure, and they verified them by the reductio ad absurdum, assuring themselves by a rigorous demonstration that every contrary supposition led infallibly to some contradiction.

In this way Archimedes proved, that the surface of a right cone is equal to the area of the circle which has its radius the mean proportional between the side of the cone and the radius of the circle of the base; and that the whole area of the sphere is quadruple one of its great circles, and that the area of any one of its zones is equal to the circumference of the great circle multiplied by the height of this zone.

It was besides by the reductio ad absurdum that the ancients extended to incommensurable quantities the ratios which they had discovered between commensurable. This theory is certainly very elegant and important: it carries with it the character of the most perfect evidence, and does not allow us to lose sight of its object: it was a method invented by the ancients; it is still very useful, since it exercises the judgment which it accustoms to the rigour of demonstration, and comprises within itself the germ of the Infinitesimal Analysis. It is true that it requires some exertion of the mind: but has not reflexion been indispensable to all who desire to advance in the knowledge of the laws of nature; and is it not necessary to acquire the habit of it betimes, provided that too much time be not sacrificed to it?

63. By attentively observing the operations of this method of Exhaustion, we see that they in fact make auxiliary quantities interpose in the investigation of the properties or of the ratios of the proposed quantities: these last are considered the extreme terms to which the first are supposed to approach continually, and the law of continuity which they follow in this approach points out the modifications by which one may pass from the known properties of these auxiliaries to the properties of the proposed quantities, which up

to this time have been unknown. It is thus that we apply the method of Exhaustion to the investigation of the properties of curves by means of inscribed and circumscribed polygons, or auxiliary systems of known quantities, which, gradually approaching to the proposed curve, teach us by the law of " continuity" which they observe in this progress, and by their analogy which becomes more and close to the curve, the nature and properties of the latter.

64. The method of Exhaustion has then essentially the same end, and follows in its operation the same principles, as the Infinitesimal Analysis. There is always the same auxiliary system of known quantities, united on one side to the sought quantities, whilst on the other there remains in this system enough that is arbitrary for us to be able at our discretion to reduce it by degrees to the proposed system: this by deduction informs us of the sought ratios. There remains then but to confirm the certainty of these ratios, and this we arrive at by " reductio ad absurdum."

65. Newton advanced this doctrine greatly towards perfection by means of his theory of prime and ultimate ratios, which makes us recognize the law of continuity in the gradual approach of the auxiliary towards the " assigned

system." By this new theory he extended the principles of the method of Exhaustion, and simplified its operations, by disencumbering it of the necessity imposed on it of always confirming by " reductio ad absurdum" the exactness of the ratios that it had discovered, and by proving that these ratios are sufficiently established by the method itself employed to obtain them. This is what Newton declares at the end of his explanation of this theory. " I have begun," he says, " by these lemmas, to avoid deducing long demonstrations by the reductio ad absurdum, according to the method of ancient geometricians."

This great man eventually advanced still farther in this science, by reducing his method of prime and ultimate ratios into a regular algorithm by his calculus of fluxions. By means of this calculus he introduced into algebraic analysis not only these prime and ultimate ratios, but even their terms taken separately, i. e. the numerator and denominator of the fraction representing either of them, by itself: a change of the greatest importance, by reason of the new methods of transformation that it supplied. To this we will return further on. But Newton does not enjoy this honour alone; he shares it with Leibnitz, who had the advantage of publishing his notation first, and who being powerfully seconded by other geometricians who immediately embraced his method, enabled it in concert with them to make more rapid strides than the calculus of fluxions could have done.

On the Method of Indivisibles.

66. Cavalerius was the precursor of those learned men to whom we are indebted for the Infinitesimal Analysis: he opened their career to them by his Geometry of Indivisibles. In the method of Indivisibles we consider lines as made up of points, surfaces as composed of lines, and solids of surfaces. These hypotheses are certainly absurd, and we must employ them with circumspection; but we must regard them as means of abbreviation, by help of which we readily and easily obtain in many cases, what we could only discover by long and troublesome processes, and, by strictly following the method of exhaustion. If, for example, it be required to prove that two pyramids of the same base and height are also of the same volume, we regard them both as composed of an infinite number of plane surfaces equally distant, which are its elements: but as these elements are severally equal, and as

their number is the same on both sides, we hence conclude, that the volumes of the pyramids which are the respective sums of these elements are equal to one another.

67. Cavalerius openly avowed that his method was only a corollary to the method of exhaustion, but owned that he could not give a rigorous demonstration of it. The great geometricians who followed him soon made themselves masters of the spirit of it; it was in great reputation with them up to the time of the discovery of the new calculations, and they placed no more account on the objections which were then raised against it, than the Bernoulli did on those which afterwards arose against the Infinitesimal Analysis. It is on this method of Indivisibles that Pascal and Roberval founded the success of their deep researches into the nature of the cycloid, and the first of these famous authors expresses himself on this subject in the following manner.

" I have made this declaration in order to shew, that every thing that is demonstrable by the true rules of Indivisibles may be demonstrated also by the rigorous method of the ancients, and thus that these methods do not differ from each other but in expression: which ought not to offend reasonable persons, when once they are shewn what is meant by it. And for this reason, I shall henceforth make no scruple of using this language of the Indivisibles, " the sum of the lines," or " the sum of the planes." I shall not hesitate to use this expression, " the sum of the ordinates," which seems, to those who do not understand the doctrine of Indivisibles, to be ungeometrical, and who consider it an error against Geometry, to call a plane an indefinite number of lines. This arises merely from their ignorance, since we understand by it no other than the sum of an indefinite number of rectangles made by each of the ordinates with each of the small equal portions of the diameter, the sum of which is certainly a plane. So that when we speak of " the sum of an indefinite number of lines," we always have respect to a certain straight line by the equal indefinite portions of which they are multiplied. Here there is certainly more said than was necessary to shew that the meaning of these terms, " the sum of the lines," " the sum of the planes," &c. are only in strict conformity with pure Geometry."

68. This passage is remarkable, not only because that the geometricians well knew how to appreciate the merit of the method of Indivisibles, but besides, because it proves that the notion of infinity in Mathematics was not strange to these

geometricians: for it is clear, from what has just been quoted from Pascal, that he attached the same signification to the word "indefinite," that we do to the word "infinite;" that he called that simply "small," which we call "infinitely small;" and that he unhesitatingly neglected these small quantities in comparison with finite quantities: for we see that Pascal regarded the trapeziums, or small portions of the area of the curve contained between consecutive ordinates, as simple rectangles, consequently neglecting the small mixtilinear triangles which have for their bases the difference of these ordinates. No one however has dared to attack Pascal for want of accuracy. Roberval constantly employs the expressions "infinite" and "infinitely small," in the sense attributed to them in the present day; and he says formally, that we must neglect infinitely small quantities, in comparison with finite quantities, and these last too when compared with infinite quantities.

They knew then from that period that the method of Indivisibles, and all those of the same kind that we can imagine, were but formulas of abbreviation, highly useful in evading the tedious method of exhaustion, without any wrong done to the exactness of its results. The geometricians who followed used the same for a long time until the differential and integral Calculus was invented. It is not then to be wondered at that Leibnitz did not apply himself to demonstrate rigorously a principle which was generally regarded as an axiom. Objections were not raised against this principle until it had been reduced into an algorithm, as if it were a source of regret that the paths of science, hitherto so difficult to tread, had all at once been cleared, and made accessible to every one.

117. Ordinary algebra shews how to find the sum of any series of terms taken in the order of natural numbers, the sum of their squares, of their cubes, &c. and this furnishes the geometrician who uses the method of indivisibles with the means of finding the area of a great number of rectilinear and curvilinear figures, and the volume of a great many solids.

Let us take for example a triangle: draw from its vertex a perpendicular to the base, divide this perpendicular into an infinite number of equal parts, and draw from each of these dividing points a straight line parallel to the base and terminated by the two other sides of the triangle. According to the principles of the geometry of indivisibles, we may consider the area of the triangle as the sum of all the parallels which are regarded as its elements: now by the property of the triangle, these right lines are proportional to their distance from the vertex: then the height being supposed to be divided into equal parts, these parallels increase in arithmetical progression, the first term of which is zero. But in every arithmetical series the first term of which is zero, the sum of all the terms is equal to the last multiplied by half the number of terms. Now here the sum of the terms is represented by the area of the triangle, the last term by the base, and the number of terms by the height. Hence the area of every triangle is equal to the product of its base and half its height.

On the method of Indeterminate Quantities.

69. It appears that Descartes, by his method of Indeterminate Quantities, touched very nearly on the Infinitesimal Analysis, or rather that the Infinitesimal Analysis is nothing more than a happy application of the method of Indeterminate Quantities. The fundamental principle of the method of Indeterminate Quantities or of Indeterminate coefficients consists in this, that if we have an equation of this form

$$\text{A} + \text{B}x + \text{C}x^3 + \text{D}x^4 + \&\text{c.} = o,$$

in which the coefficients A, B, C, &c. are constants, and x a variable quantity, which may be supposed as small as we please, it is absolutely necessary that each of these coefficients taken separately be equal to zero; i. e.

$$A = o,\ B = o,\ C = o,\ \&c.$$

whatever the number of the terms of this equation may be. In fact, since we are at liberty to suppose x as small as we please, we may also make as small as we please the sum of all the terms which have x for a factor, that is to say, the sum of all the terms which follow the first. Hence this first term differs as little as we please from o; but A being constant, cannot differ as little as we please from o, for then it would be a variable; hence it can only be o: there remains therefore

$$Bx + Cx^2 + Dx^3 + \&c. = o;$$

dividing by x,

$$B + Cx + Dx^2 + \&c. = o;$$

whence we deduce $B = o$ for the same reason already given to prove $A = o$; the same reason will similarly prove $C = o$, $D = o$, &c.

70. In the next place, let there be an equation of two terms only,

$$A + Bx = o,$$

in which let the first term be constant, and the second susceptible of being made as small as we please: this equation cannot exist after what has been said above, unless the terms A and Bx be each of them separately equal to zero. Hence we can establish as a general principle and as an immediate corollary from the method of indeterminate quantities, that " if the sum or the difference of two proposed quantities be equal to zero, and if one of the two may be supposed as small as we please, whilst the other includes no arbitrary part in it, these two proposed quantities will be each severally equal to zero."

71. This principle is of itself able to solve by common Algebra all questions arising from the Infinitesimal Analysis. The respective operations of both methods, simplified as they ought to be, are absolutely the same, all the difference is in the manner of regarding the question: the quantities, which in the one we neglect as infinitely small, in the other we imply, at the same time that they are considered as infinite, since it has been shewn that they ought to eliminate themselves, i. e. mutually destroy one another in the result of the Calculus.

It is easily observed, that this result can only be an equation of two terms, each of which is severally equal to zero: hence we may drop beforehand in the course of the calculation all the terms, which have reference to that one of these two equations, of which we do not wish to make use.

72. By way of example, let us attempt to prove that the area of a circle is equal to the product of its circumference by the half of its radius; i. e. calling R the radius, π the ratio of the circumference to the same radius, and consequently πR this circumference, S the surface of the circle, we have

$$S = \frac{1}{2}\pi R^2.$$

To do this, inscribe a regular polygon in the circle, then double successively the number of these sides until the area of this polygon differs as little as we please from the area of the circle. At the same time the perimeter of the polygon will differ as little as we please from the circumference πR; and the diagonal as little as we please from the diameter. Hence the area will also differ as little as we please from $\frac{1}{2}\pi R^2$; hence if we make $S = \frac{1}{2}\pi R^2 + \phi$, the quantity φ, unless it equal zero, may be considered as small as we please. Next put the equation under the form,

$$(S - \frac{1}{2}\pi R^2) - \varphi = 0,$$

an equation of two terms, the first of which comprises in it no arbitrary quantity, and the second, on the other hand, may be supposed as small as we please; hence, by the theory of indeterminate quantities, each of these terms is severally equal to o, and

$$S - \tfrac{1}{2}\pi R^2 = o, \text{ or } S = \pi R^2.$$

On the method of Prime and Ultimate Ratios, or of Limits.

73. The method of prime and ultimate ratios, or of limits, likewise derives its origin from the method of exhaustion; and is only, properly speaking, a development and simplification of this latter. It is to Newton that we owe this useful improvement of it, and in his book of the Principia must we look for information respecting it: our end will be gained by giving a slight sketch of it.

When any two quantities are supposed to approach continually to each other, so that their ratio or quotient differs less and less gradually, and as little as we please from unity, these two quantities are said to have their ultimate ratio a ratio of equality.

Generally, when we suppose that different quantities mutually and in equal portions of time approach other quantities which are considered as fixed, until they all at the same time differ as little as we please, the ratios existing between these fixed quantities are the ultimate ratios of those which are supposed to approach respectively and in equal portions of time; and these quantities themselves are called limits or ultimate ratios of those which thus approach one another.

These ultimate values and ratios are also called prime values and ratios of the quantities to which they refer, according as we consider the variables approaching to, or retiring from, the quantities which are considered fixed, and serve for their limit.

74. These limits or quantities, considered as fixed quantities, may nevertheless be variable, as, for example, the coordinates of a curve would be: that is to say, that they cannot be given by the conditions of the question, but merely determined by the subsequent hypothesis on which the calculation is established. Thus, for example, although the coordinates of a curve are amongst the quantities which we call variable, since they are not amongst the given quantities, if it be proposed to solve a question relative to the curve spoken of, as to draw a tangent to it, it will be necessary, in order to establish my system of argument and calculation, to begin by assigning

determinate values to these coordinates, and to consider them fixed unto the end of our calculation. Now these quantities considered as fixed are, as well as the given quantities of the question, amongst those which we call limits.

75. These limits are no other than the quantities of which the ratio is sought; those which are supposed to approach gradually are only auxiliary quantities, which we interpose to facilitate the expression of the conditions of the equation, but which cannot remain in the calculation, and which we must necessarily eliminate in order to obtain the desired result: they are consequently of the class of those which we have named "unassigned," whilst their limits or ultimate values are the sought quantities which we call "assigned." Thus we see the analogy which must exist between the theory of prime and ultimate ratios and the Infinitesimal Analysis. For that which in the latter we call an infinitely small quantity, is evidently nothing more, according to the definition which we have given, (14,) than the difference between any quantity and its limit, or, if you please, a quantity the limit of which is *o*: and quantities which have for their ultimate ratio a ratio of equality, are

nothing more than those which in the Infinitesimal Analysis are called quantities differing infinitely little from each other.

76. We see, besides, that the notion of an infinitely small quantity is not less clear than that of the limit, since it is nothing else than the difference of this same limit and the quantity of which it expresses the ultimate value. But the difference between what we properly call the method of limits and the Infinitesimal method consists in this, that in the first we admit only the limits themselves, which are always "assigned" quantities: whilst in the Infinitesimal method we admit also "unassigned" quantities, which are supposed to approach continually to equality, and the differences between these "unassigned" quantities and their limits; which gives to the Infinitesimal Analysis more means of varying its expressions and algebraic transformations, without at all influencing the rigour of the operation.

77. The advantage which the Infinitesimal method thus procures to itself, makes it susceptible of a still farther degree of perfection, viz. of being reduced into a peculiar algorithm. For these differences between the "unassigned" quantities and their limits are what we distin-

guish by the name of "differentials" of these same limits, and the simplifications which the admission of these quantities into the Calculus occasions are what supplies the Infinitesimal Analysis with such powerful means.

Nevertheless, the method of limits, fettered as it is by the principles according to which it omits introducing into the Calculus the auxiliary quantities of which these limits are only the ultimate values, far surpasses the simple method of exhaustion in facilitating the calculations; since it at any rate gets rid of the "reductio ad absurdum" for every particular case; an operation more laborious than any others which enter into the method of exhaustion; whilst in the other, in order to establish the equality of any two quantities whatever, it is sufficient to prove that they are all limits of one third quantity.

78. No distinction is to be made between the doctrine of limits and that of prime and ultimate ratios. Newton has made none: he uses indifferently the name of the limit of a quantity, or the ultimate value of that quantity:—the limit of the ratio of two quantities, or the ultimate ratio of these two quantities. This observation has been made, because there are persons who vaguely think that there is some difference be-

tween the method of limits, such as a D'Alembert has explained it in the article "Differéntiel" of the Encyclopédie, and the method of prime and ultimate ratios, such as Newton explained it in his book of the Principia. It is absolutely the same thing, and D'Alembert declares positively in this article that he is only the interpreter of Newton.

On the Method of Fluxions.

79. Newton considers a curve to be generated by the uniform movement of a point: he decomposes the constant velocity of this point at each moment into two others, the one parallel to the axis of the abscissa, and the other parallel to the axis of the ordinates. These velocities are what he calls fluxions of these coordinates, whilst the arbitrary velocity of the point which describes the curve is the fluxion of the arc described. Reciprocally this arc is called the fluent of the velocity with which it is described by the moving point, the corresponding abscissa is called the fluent of the velocity of this point calculated in the direction of this abscissa, and the ordinate is called the fluent of the velocity of this same point reckoned in the direction of this ordinate.

Since the fluxion of the arc is supposed constant, it is evident that unless the describing point moves in a straight line, the fluxions of the abscissa and of the ordinate will be variable, and that their ratio at each instant will depend on the nature of the curve, i. e. on the ratio itself of these coordinates. Reciprocally, the ratio of the coordinates necessarily depends on that which exists at each instant between the fluxions of these coordinates. We may consequently enquire, what is the method of finding the ratio which exists between the fluxions, when we know that which exists between the coordinates: and conversely, what is the way to discover the ratio between the coordinates, when we know that which exists between the fluxions separately or in combination with the coordinates themselves. The first part of this problem is what is called the method of fluxions, and the second the inverse method of fluxions.

80. But these primary ideas may be generalized; for whilst the describing point runs through the curve, not only do the abscissa and ordinate change, but besides these, the subtangent, the normal, the radius of curvature, &c.: that is to say, that these quantities increase or decrease more or less rapidly, as well as the co-

ordinates themselves. All these quantities then have fluxions, the ratios of which are equally determined by the movement of the point which uniformly describes the curve: thus these quantities themselves are the fluents. Now it is the art of determining the ratios of all these fluents by the intervention of their fluxions, (used as auxiliaries,) which is called the direct and inverse method of fluxions, or the method of fluxions and fluents.

81. The method of fluxions admits, as we have observed, finite quantities only, since these fluxions are only velocities which are finite quantities. We may even take these velocities respectively with which the coordinates increase for coordinates of a new curve, which will also have their fluxions, and these will in a similar way be finite quantities: and these last again may be taken for coordinates of a third curve, and so on without introducing into the calculation any other than finite quantities.

82. There is a primary fluxion chosen arbitrarily, but which when once adopted regulates all the others: we may choose which we please; we have taken the absolute velocity of the generating point which we regarded as uniform; but we may with the same reason suppose it to be the

velocity in the direction of the abscissa, or any other which is uniform, and serves as a term of comparison.

83. The method of fluxions and fluents springs naturally out of that of prime and ultimate ratios: for the variable velocity of a point is not the path described by this point in a given time, divided by this time, but the prime or ultimate ratio of this ratio; that is to say, the quantity by which this ratio approaches more and more nearly, according as this time is supposed to diminish.

84. This remark has been used to raise an objection to the method of fluxions; for, say the objectors, it is introducing into Geometry, which is a part of pure Mathematics, the notion of velocity, which enters only into mixed Mathematics; and it is to define an idea which ought to be simple, by another which is complex. But this objection is frivolous enough: for the real object of consideration is to ascertain whether the theory is more easily apprehended in this manner than in any other. The classification which we make of sciences is quite arbitrary. We place Geometry before Mechanics in the order of simplicity: but the transcendent parts of the former are much more abstract than the elementary parts of

the second: and, as Lagrange says, "Every one has or thinks he has a clear idea of velocity;" it is not acting in opposition to the spirit of Mathematics to define fluxions by velocities.

85. We have just seen, that the velocities which we call fluxions are the ultimate ratios of the spaces described, and of the times employed to describe them: but if we compare two of these velocities or fluxions, (for example, the fluxion of the absciss with that of the ordinate,) these fluxions will themselves have a ratio to each other, which is only the limit of the ratio of the differentials of these coordinates.

Thus the method of fluxions is nothing else, as we regard it, than the Infinitesimal method: and consequently it is the method of exhaustion regarded in another point of view; and we readily perceive the link which unites all these methods with each other.

84. The operations of the method of fluxions only differ from those of the Infinitesimal Analysis in notation. Instead of the characteristic d which is employed in the latter, we make a dot over the letters in the method of fluxions: i. e. the fluxion of the variable fluent x is represented by $\dot{x}$, but with this distinction, that $\dot{x}$ represents a finite quantity, which is the velocity of the

generating point in the direction of the abscissa, whilst dx in the differential calculus represents an infinitely small quantity, which is the increment of this same abscissa. In the same way, if we imagine a new curve the coordinates of which are the respective fluxions of x and y, the fluxions of these new coordinates will be the fluxions of fluxions, and will become according to the notatation explained $\ddot{x}$, $\ddot{y}$; and $\ddot{x}$, $\ddot{y}$, and will still be finite quantities, whilst the differentials d^2x, d^2y corresponding to them in infinitesimal method are infinitely small quantities of the second order ; and so on.

85. It is not our province to decide between Newton and Leibnitz respecting the priority of their invention. It appears to me that the metaphysical principles of the one of these methods is so different from that of the other, that it is more than probable that each invented his own. The history of Mathematics is full of similar examples: truth is a single point, and science constantly makes towards it: no sooner is it perceived, than each method hastens to occupy the path which it has made for itself. We must remember, that at the time of Newton and of Leibnitz, many ideas analogous to those of these two great men were transmitted by

the works of the learned throughout the world. It was indeed ripe fruit. Cavalerius, Fermat, Pascal, had subjected infinitely small quantities to calculation: Descartes had found out the method of Indeterminate quantities: Roberval had invented the decomposition of the velocity of a point describing a curve into two other velocities parallel to the two coordinates: Barrow had considered curves as polygons of an infinite number of sides: Wallis had pointed out the calculation of series: it remained but to reduce all these discoveries of the same kind to one uniform method by a notation. Is it not more natural to suppose, that Newton and Leibnitz each discovered his own by very opposite reasoning, than to conclude that either one of these two great men, already celebrated for so many other reasons, has been a plagiarist to the other?

On the Calculus of Vanishing Quantities.

86. The greater part of the learned, in order to reconcile the simplicity of the method of Leibnitz with geometrical rigour, have recourse to considering infinitely small quantities as ab-

solutely zero. The metaphysical principles of the Infinitesimal Calculus is exhibited with great clearness under this point of view in Euler's preface of the differential Calculus: " The differential Calculus," says that great geometrician, " is the art of finding the vanishing increments that any functions acquire, when we attribute to the variable quantity of which they are functions a vanishing increment."

Newton had already admitted into his Principia the notion of vanishing quantities. " We must," he says, " understand by the ultimate ratio of vanishing quantities the ratio which diminishing quantities have to each other, not before they vanish, nor after they have vanished, but at the very moment of their vanishing."

D'Alembert rejects this explanation, although he in other respects adopts altogether the doctrine of Newton on the limits or prime and ultimate ratios of quantities.

" This method," says Lagrange, " has the great inconvenience of considering quantities in the state when, if I may express myself so, they are ceasing to be quantities; for although we always readily understand the ratio of two quantities so long as they remain finite, this ratio no longer offers to the mind a clear and precise idea the moment that its terms become both zero at once.

It appears nevertheless, that infinitely small quantities being variables, nothing prevents our assigning to them the value *o* as well as any other. It is true that then their ratio is $\frac{o}{o}$, which may as easily be supposed *a* or *b* as any other quantity whatever.

87. The consideration of these vanishing quantities would then be nearly useless, if we confined ourselves to treating them in the Calculus as quantities absolutely zero: for they would present nothing further than the ratio of *o* to *o*, which is no more equal to 2 than to 3, or to any other quantity. But we must not fail to observe, that here these zeros have peculiar properties, as ultimate values of indefinitely decreasing quantities of which they are the limits; and that we do not give them the peculiar denomination of vanishing quantities before we declare, that of all the ratios which they can have, we wish to consider, and to introduce into the combinations, those only which are assigned to them by the law of continuity, when we imagine the system of auxiliary quantities approaching by insensible degrees to the system

of " assigned" quantities: and this is exactly what Newton means when he says, that vanishing quantities are quantities taken not before they vanish, nor after they have vanished, but at the very moment of their vanishing.

In the case treated of above (9), so long as RS does not coincide with MP, the fraction $\frac{\text{MZ}}{\text{RS}}$ is greater than $\frac{\text{TP}}{y}$; these two fractions become equal only when MZ and RZ become equal to zero. It is true that then $\frac{\text{MZ}}{\text{RZ}}$ is as much equal to any other quantity as to $\frac{\text{TP}}{y}$, since $\frac{o}{o}$ is an absolutely arbitrary quantity: but amongst the different values that we may assign to $\frac{\text{MZ}}{\text{RZ}}$, $\frac{\text{TP}}{y}$ is the only one which is subjected to the law of continuity, and determined by it: for if we were to construct a curve of which the abscissa was an indefinitely small quantity MZ, and the ordinate proportional to $\frac{\text{MZ}}{\text{RZ}}$, that which would answer to the abscissa zero would be represented by $\frac{\text{TP}}{y}$, and not by any arbitrary quantity; now this is what distinguishes the quantities, which have been called vanishing, from those which are really zero.

Thus, although in general we say

$$o = 2 \times o = 3 \times o = 4 \times o, \text{ \&c.}$$

we cannot say of a vanishing quantity, such as MZ,

$$\text{MZ} = 2\ \text{MZ} = 3\ \text{MZ} = 4\ \text{MZ} = \text{\&c.}$$

for the law of continuity cannot assign any other ratio than that of equality between MZ and MZ, nor any other relation than that of identity.

88. We have found, that by introducing infinitely small quantities into the Calculus, and by neglecting them in comparison with finite quantities, the equations become imperfect, and that the errors to which they have given rise are compensated for only in the sought result.

We may however avoid this inconvenience if we wish it by the method of vanishing quantities, which being nothing more than the ultimate values of indefinitely small corresponding quantities, may like every other value be attributed to these indefinitely small quantities; and which being on the other hand absolutely zero, may be neglected when found added to any real magnitudes without the calculation losing any thing of its exactness.

89. We may then regard the Infinitesimal

Analysis in two points of view: by considering the infinitely small quantities either as real quantities, or as absolutely zero. In the former case, Infinitesimal Analysis is nothing more than a calculation of a compensation of errors: and in the second it is the art of comparing vanishing quantities together and with others, in order to deduce from these comparisons the ratios, whatsoever they may be, which exist between the proposed quantities. These quantities, as equal to zero, ought to be overlooked in the calculation when they are found in addition with any real quantity, or when they are subtracted from them: but nevertheless they have, as we have just seen, ratios very interesting to discover, and such as are determined by the law of continuity, to which the system of auxiliary quantities is subject in its changes. Now in order to readily apprehend this law of continuity, we may easily observe, that we are obliged to consider the quantities in question, at some distance from the term when they vanish altogether, if they would not present the indefinite ratio of *o* to *o:* but this distance is arbitrary, and has no other object than to enable us to judge the more easily of the ratios which exist between vanishing quantities: these are the ratios which we have in view, whilst we regard the infinitely small quantities as absolutely zero, and not those which exist between the quantities which have not yet arrived at the term of their annihilation. These last, which have been called infinitely small, are never intended themselves to enter into the calculation, regarded under the point of view of which we are at present speaking, but are only employed to assist the imagination, and to point out the law of continuity which determines any ratio whatever of the vanishing quantities to which they correspond.

Thus, according to this hypothesis in the proportion (*Fig.* 1.), MZ : RZ : : TP : MP, the quantities represented by MZ and RZ are supposed equal to zero: but as it is their ratio that we want, in order to perceive its equality to $\frac{\text{TP}}{\text{MP}}$, we must consider the indefinitely small quantities which answer to these zeros, not with a view to introducing them into the Calculus, but to bring in under the denomination of MZ and of RZ the evanescent quantities which are their ultimate values.

90. These expressions MZ, RZ represent no quantities here, and we employ them under the forms RZ, MZ, rather than under the common form of *o*, because if we used them under this

latter form, we could no longer distinguish their different origins in the operations where they are intermixed, that is to say, that there are different quantities indefinitely small which correspond to them. Now this last consideration, at least so far as the mind goes, is requisite in order to apprehend the law of continuity which determines the ratio of the evanescent quantities which they have as limits, and it is consequently important not to lose sight of them, and to characterize them by expressions which may prevent their being mistaken.

91. The evanescent quantities which are the subject of the Infinitesimal Analysis, regarded in this point of view, are indeed the creatures of reason: but that does not prevent them from having mathematical properties, nor us from being able to compare them, as well as we compare imaginary quantities, which exist only in the imagination. No one questions the exactness of the results which we obtain by the Calculus of imaginary quantities, although they are but algebraic forms, and the hieroglyphics of unreal quantities: a fortiori we ought not to exclude evanescent quantities, which are nevertheless limits of real quantities, and approach nearly (if we may use the expression) to existence. What signifies

it whether these quantities be ideal or no, if their ratios are not, and if these ratios are what alone interest us? We have it then entirely in our power, when submitting the quantities which we have called Infinitesimal to the Calculus, to regard these quantities as real or as absolutely zero: and the difference between these two manners of regarding the subject consists in this, that by taking these quantities for zero, the propositions (equations and results of all kinds) remain exact and rigorous during all the calculation, but have reference to quantities that are the creatures of reason, and express the ratios existing between quantities which themselves do not exist: whilst by considering the infinitely small quantities as something real, the propositions (equations and results of any kind whatever) have for their subject real quantities; but these propositions are false, or rather they are imperfect, and become exact at last only by a compensation of errors: which compensation, however, is a necessary and certain consequence of the operations of the Calculus.

92. The metaphysical principles which have been exhibited easily furnish answers to all objections that are brought against the Infinitesimal Analysis, the principle of which many geometri-

cians have thought false, and liable to lead into errors: but they have been overwhelmed by the number of prodigies, and the great truths which sprung out of this principle in abundance. These objections may be reduced to this: either the quantities which we call infinitely small are absolutely zero, or they are not: for it is absurd to suppose that there exist things holding a middle rank between quantity and zero. Now if they are absolutely nothing, the comparison of them leads to nothing, since the ratio of *o* to *o* is not more that of *a* to *b* than any other quantity; and if they are real quantities, we cannot without error treat them as zero, as the laws of Infinitesimal Analysis prescribe. The answer is simple: far from being unable to consider the infinitely small quantities as neither real nor as nothing, we may on the contrary say, that we can at our pleasure regard them as zero or as real quantities; for those who wish to look upon them as zero, can answer that the quantities which they call infinitely small are not any zeros whatsoever, but zeros assigned by a law of continuity which determines their ratio: that amongst all the ratios of which these quantities are susceptible as zero, they consider those only which are determined by the law of continuity; and, lastly, that these ratios are not vague and arbitrary, since this law of continuity never assigns, for example, several ratios to the differentials of the absciss and the ordinate of a curve when these differentials vanish, but one only, namely, that of the subtangent to the ordinate. On the other hand, such as regard the infinitely small quantities as real quantities may answer, that what they call infinitely small is an arbitrary magnitude, and susceptible of being supposed as small as we please, without affecting the proposed quantities; that, in the next place, without regarding it zero, we may nevertheless use it as such, without any error resulting, since this error if it existed would be arbitrary like the quantity which gave rise to it. Now it is evident, that a similar error could only exist between such quantities as were, at least in one case, arbitrary. Hence, when we have arrived at a result which no longer contains them, and which expresses any one ratio whatever between the given quantities and those which are determined by the conditions of the problem, we may be sure that this result is exact, and that consequently the errors, which must needs have been made in expressing these conditions, have been compensated for, and disappeared, by a necessary and certain consequence of the operations in the Calculus.

93. Other geometricians, apparently embarrassed by the objection which has just been dispatched, have endeavoured simply to prove, that the method of limits, the operations of which are rigorously exact in all points, must necessarily lead to the same results as the Infinitesimal Analysis. But whilst we allow that the principle of this method is very clear, we cannot hide from ourselves that they only elude without solving the difficulty; that the method of limits does not lead to the Infinitesimal Analysis but by a difficult and tortuous path; and lastly, that this method, far from being the same as that of the Calculus of Infinitesimals, is on the contrary merely the art of passing it by, and of supplying their place by common Algebra: in which we might succeed better by the method of indeterminate quantities.

94. It follows from what we have advanced, that we may at discretion consider infinitely small quantities either as absolutely zero, or as real quantities: one motive, however, would make me prefer this last method of regarding the Infinitesimal Analysis: it is this, that those

who consider it thus, appear to treat the question more generally than the rest. For these last, in assigning the value *o* to infinitely small quantities, go through an ineffectual operation: they seem to look on this termination as necessary, and to think that without it they could not obtain what they seek: which is by no means the case, since these quantities may all be eliminated without conditions, i. e. without their assigning to them any determinate value, and not that of *o* more than any other. The question consequently appears to be solved in a more general way, when we leave undetermined such quantities as we are not obliged to determine.

On the theory of Analytic or Derived Functions.

95. None of the methods practised or proposed up to the time of Lagrange, in order to supply the place of the method of exhaustion, and to bring it into a regular notation, appeared to him to unite the exactness and simplicity requisite for Mathematical science. He was, however, of opinion, that it was not impossible to attain this important end, and his researches with this view have occasioned the great work

published by him under the title of "Theorie des fonctions analytiques, contenant les principes du calcul différentiel, degagés de toute consideration d'infiniment petits, d'évanouissants, de limites et de fluxions et reduits a l'analyse algebrique des quantités finies." Lagrange moreover produced another work on the same subject, called, "Leçons sur de calcul des fonctions," which is a commentary on the first, and a supplement to it. These writings are marked with the stamp of that original and profound genius, to which we were already indebted for the Calculus of Variations and the Analytic Mechanics; but as they must be in the hands of all who are desirous of entering deeply into the science of the Calculus, they will be noticed but briefly.

With a view to preserve in the whole course of his operations that rigorous exactness from which Lagrange never deviated, by making use of differentials under another denomination, and with a different notation, he considers them as finite indeterminate quantities. Consequently he does not neglect any term, and finds his differentials as we do in the Calculus of finite differences. He arrives at this by Taylor's Theorem, which he makes the base of his system, and which he proves directly by the ordinary analysis, whilst before his time it had been proved only by aid of the Differential Calculus.

The author thus contrived to express the conditions of every proposed question by rigorously exact equations. These equations must have been with more difficulty established, and more complicated than those which we get by the common operations of the Infinitesimal Analysis; when we overlook the infinitely small quantities in comparison with the finite quantities. But as both one and the other of these equations can never lead but to the same results, we perceive that there must be means of simplification for the former which reduce them to the latter. And this is in fact the case: the inventor by a course of ingenious transformations at length disengages his Calculus of every thing that perplexed it. And in this way these equations return of themselves, and without our neglecting any thing in the course of the operations, to the simple state of those which we might have obtained by the ordinary process of the Infinitesimal Analysis.

96. Although Lagrange takes his differentials as if they were finite differences, they have a

character which essentially distinguishes them from these last: it is this, that they continue always indeterminate, so that we can, during the whole operation, make them as small as we please without at all changing the value of the sought quantities: this furnishes means of elimination which never belong to the ordinary calculus of finite differences, in which these differences are fixed.

96. It is not difficult to make out the analogy between the theory of analytic functions, that of the common infinitesimal analysis, and the method of indeterminate quantities already spoken of. The differences taken without neglecting any thing, as done in the theory of analytic functions by Taylor's formula, are series. Now, as Lagrange himself observes, all the problems which require the differential calculus for their solution, depend solely on the first term of these series, and all the methods have but one end in view, viz. to detach this first term from the rest of the series. The common differential calculus effects its object immediately by neglecting from the first all the rest, as if they were zero: in the method of indeterminate quantities we merely imply them, as though they must necessarily destroy each other in the result of

the calculation: lastly, in the theory of functions we make them actually enter into the expression of the conditions of the problem, and we eventually eliminate them by transformations founded on this property, that all the terms have for a common factor an increment of the variable, which we may suppose as small as we please, whilst the first term of the series is independent of it. This evidently recurs to the method of indeterminate quantities, by which we are led to equations in which each of the terms taken separately are equal to zero.

98. The real obstacle to so clear a method is the novelty of the notation, for the sake of which it would be necessary to abandon that which long custom has *consecrated*, and in *unison* with which all the original works of a century were written. Thus, for example, it would be necessary to remodel all the collections of the Academy, all Euler's works, and those of Lagrange himself. This was his own idea, when he published the new edition of his " Mecanique Analytique:" he never employed his own notation, and on this subject he expresses himself in the following manner in his preface to the above work.

" We have retained the common notation of the differential calculus, since it answers to the system of infinitely small quantities adopted in this treatise. When once we have rightly conceived the spirit of this system, and have been convinced of the exactness of its results by the geometrical method of prime and ultimate ratios, or by the analytic method of derived functions, we may use the infinitely small quantities as a safe and convenient means of abridging and simplifying demonstrations."

99. Some years ago, John Landen, a learned English geometrician, attempted, not altogether without success, to reduce the Infinitesimal Analysis to common algebra: Lagrange, who took pleasure in bringing forward the merit of others, since he felt himself rich enough in his own discoveries, speaks of John Landen in terms of commendation to the following effect.

" To obviate these difficulties, a skilful English geometrician, who made important discoveries in analysis, proposed lately to substitute instead of the method of fluxions, hitherto followed scrupulously by all English geometricians, another method purely analytic, and analogous to the differential method, but in which, instead of using merely the infinitely small differences of variable quantities, we employ different values of these quantities, which are subsequently made equal, after having by division got rid of the factor, which this state of equality made equal to zero. By this way we do indeed avoid the infinitely small and evanescent quantities, but the operations and applications of this calculus are embarrassing and unnatural; and we must allow, that this manner of rendering the calculus more rigorous in its principles, makes it lose its principal advantages, simplicity, and facility."

The theory of analytic functions itself is not perhaps exempt from some of these inconveniences: it belongs to such as have acquired the habit of using it to decide upon this.

CONCLUSION.

100. The different methods which we have described in this work are, properly speaking, one and the same method, presented under different points of view: it is always the method of exhaustion of the ancients more or less simplified, more or less happily applied to the wants of the calculus, and at last reduced to a regular notation. But this notation is of great importance, it is to the method of exhaustion what common algebra is to synthesis; the ancients knew only of synthesis and the method of exhaustion, which is itself but a branch of synthesis: the moderns by discovering common algebra and the Infinitesimal Analysis have gained immense advantages. It is a means by which they abridge and facilitate the operations of the mind, by reducing them, if we may so express ourselves, to mechanical trouble. The symbols are not only what writing is to the thoughts a means of describing and of fixing them; they re-act upon the thoughts themselves, they direct them up to a certain point, and it is sufficient to arrange them on paper according to certain very simple rules, in order to arrive infallibly at new truths.

The algebraic symbols do more, they introduce into combinations purely imaginary forms, fictitious beings, which can neither exist nor even be conceived, and which notwithstanding are of use. We employ them as auxiliaries or terms of comparison, to facilitate the approximation to real quantities which are unknown, and then we eliminate them by transformations which themselves are only manual operations. This wonderful instrument of the pure sciences could not be produced but by reiterated attempts of the greatest geniuses, and perhaps by some fortunate chances. But we must not forget, that it is but an instrument formed to aid the imagination, and not to weaken the springs of it; it is always an indirect means of obviating the weakness of our minds: we must employ it only with reluctance, and with a view of overcoming difficulties or of generalizing questions; and to have recourse to it unnecessarily, as in the first elements of science, where it causes more embarrassment than real assistance, is clearly an abuse of it.

Far from using the analysis to establish elementary truths, we ought to disengage from them every thing which hinders us from perceiving them as distinctly as possible, and from recurring to the path which leads us to them. As for those who succeed in shewing us almost intuitively such results as we had not arrived at before them, but by the aid of a complicated analysis, do they not occasion as much pleasure to us as surprise, provided always that this is done in a simple manner, and without increasing the difficulties?

The principles of common algebra are much less clear and less firmly established than those of the Infinitesimal Analysis, in that which distinguishes the latter from the former. The metaphysical principle of the rule for signs is much more difficult to discover than that of infinitely small quantities. This rule has never been demonstrated in a satisfactory manner: it does not appear that it can be, and nevertheless it serves as the foundation of all algebra. What then would be gained by substituting this latter for the Infinitesimal Analysis, (since the operations of the former are much more complicated than those of the second,) in order to attain those objects which naturally spring from the latter? The analytic expression of an object can never be so clear as the immediate perception of the object itself: it is looking at a thing reflected by a mirror, which we might see directly. This analytic expression may be embarrassed by imaginary forms, or point out operations to be performed of which we are incapable. To define an object of sense by similar expressions is not only an useless employment of an indirect method, but it is representing a thing clear of itself by a symbol which is much less so. The following is a very remarkable passage from Euler on this subject, who was the first analyst of the last century. (Memoire de l'Academie de Berlin, année 1754.)

"There are persons," he says, "who maintain, that geometry and analysis do not require much reasoning. They imagine, that the rules which these sciences prescribe to us already comprise the knowledge necessary for arriving at the solution, and that we have only to perform these operations in conformity with these rules, without troubling one's self with the reasoning on which these rules are founded. This opinion, if it were well founded, would be contrary to the almost universal sentiment; namely, that geometry and analysis are the fittest means of employing the mind, and of exercising the reasoning faculties. Although these people have a smattering of mathematics, they can have applied themselves but very little to the solution of difficult problems; for they would have perceived, that the mere application of prescribed rules is a very weak auxiliary in solving this sort of problems; and that it is necessary, in the first place, to examine attentively all the circumstances of the problem, and previously to employing these rules, to draw many arguments, because these contain the rest of the reasoning which we with difficulty observe whilst pursuing the course of the calculation. It is this necessary preparative, before having recourse to the calculus, which very often requires a longer train of arguments than perhaps any other science ever did. And in this we are possessed of the great advantage of being satisfied of their propriety, whilst in other sciences we are frequently obliged to stop short at unconvincing arguments. But the calculus itself, although analysis prescribes to it its rules, ought to be sustained by solid reasoning, in the absence of which we run the risk of committing errors every moment. The geometrician then finds on all sides opportunities of exercising his mind by reasoning, which ought to lead to the solution of all the difficult problems which he undertakes; and unless he be on his guard, there is a chance of his coming to false conclusions."

The rule, as it appears to us, ought to be always to take the simplest method; and, when the difficulties are equal, to take the most perspicuous: no method should be used exclusively. Thus, to return to our subject, amongst the methods spoken of above, it is necessary, with a view to the habitual employment of it, to choose that which effects its end in general by the shortest and easiest way, but not so as to reject any of the others, since they are fine speculations for the mind; and besides, there is not one amongst them, perhaps, which may not lead to some truth hitherto unknown, or procure in certain cases an unobserved result, or a solution more elegant than any other. But amongst all these methods deriving their origin in common from the method of exhaustion of the ancients, which is that which offers most advantages for habitual employment? It seems generally agreed, that it is the Analysis of Leibnitz. The works of Descartes, of Pascal, Fermat, Huyghens, Barrow, Roberval, Wallis, and particularly those of Newton, prove, that for a long time they had touched on this great discovery when it was announced by Leibnitz; and it appears that there is no one amongst these illustrious geometricians who would not have done it, if he had suspected that he could make a great discovery in this quarter; which is as much as to say, that there is not one amongst them who would not have found means of reducing the methods of exhaustion to notation, if the idea had struck them of doing so, and they had foreseen all the fertility with which a similar method would one day be invested. Perhaps too, amongst the several notations invented by so many original geniuses, there would have been found some which would have obtained the preference over that of Leibnitz, which habit has endeared to us not less than the precious and costly labours, which in the present day are dressed up in the forms of this notation.

"We may," says Lagrange, "consider Fermat as the first inventor of these new calculations. Barrow conceived the substitution of real but infinitely small quantities for quantities which, according to Fermat, ought to be taken at zero; and in 1674 he produced his method of tangents, where he considers the curve as a polygon of an infinite number of sides: but this calculus was but sketched out; for it applied to rational expressions only. It remained still to find out a simple and general notation applicable to all sorts of expressions, by which one might pass immediately, and without any reduction, from algebraic formulas to their differentials: this Leibnitz effected ten years subsequently. It appeared that Newton had arrived at the same method of shortening the calculus for differentiation during the same time, or a little before[a]. But it is in the formation of differential equations, and in their integration, that the great merit and principal force of the new calculations consist; and on this point it seems that the glory of the discovery is due to Leibnitz almost entirely, and for the rest to the two Bernoullis.

It would appear very difficult now to quit the route which has been opened to us by these illustrious geometricians, to take up suddenly a new way of looking at it, a new notation and new expressions. Lagrange himself acknowledged, as we have already seen, that the method of Infinitesimals, such as we employ it in the present day, is a certain means of abbreviation and simplification; and he thought himself bound to use it in his new edition of his Analytic Mechanics, in preference to those which he had just proposed in his "Theorie des fonctions Analytiques." We are hence at liberty to presume, that this latter work was but as it were a means of assembling under one point of view many analytical contrivances, which in the course of his labours had presented themselves, and with a

view to exhibit systematically the great resources of his genius in calculation.

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[a] See Lardner's Introduction to Algebraic Geometry.

It has been remarked many times by this profound thinker, that the real secret of analysis consists in the art of making one's self acquainted with the different degrees of indetermination, of which the quantity is capable; an idea which always struck us as forcible, and which occasioned our regarding the method of indeterminate quantities of Descartes as the most important corollary to the method of exhaustion.

In all branches of analysis taken generally, we observe that these operations are always founded on the different degrees of indetermination in the quantities compared by it. An abstract number is less determinate than a concrete, since the latter determines not only the quantity but the quality of the object submitted to the calculus. Algebraic quantities are more indeterminate than abstract numbers, because they do not specify the quantity. Amongst these last, variables are more indeterminate than constants, since the latter are considered fixed for a longer period in the calculation. Infinitesimal quantities are more indeterminate than simple variables, since they still continue susceptible of change, even when it has already been agreed on to consider the

others as fixed. Lastly, the variations are more indeterminate than the simple differentials, since the latter are restricted to varying according to a given law, instead of which the law according to which the others change is arbitrary. This gradation in the different degrees of indetermination is endless, and it is in this union of quantities more or less defined, more or less arbitrary, that is founded the fertile principle of the general method of indeterminate quantities, of which the Infinitesimal Calculus is in truth but a fortunate application.

These quantities, on one hand united to the conditions of the question, whilst on the other we are free to attribute to them greater or less values, these quantities, which are in some sort arbitrary, make us observe the necessity of drawing that distinction which we have made between "assigned" and "unassigned" quantities; a distinction which is not the same as that between constants and variables; for "assigned" quantities comprise under their term the unknown constants and variables, and their functions; that is to say, all those which can enter into the results of the Calculus, whilst "unassigned" quantities are necessarily excluded from them. The latter can enter then merely as auxiliaries, serving

only to facilitate the expression of the conditions of the problem: after which all the energies of the calculator must be employed with a view to their elimination, which is indispensable in every case, and always shews, when it is completed, that this calculation loses from that time forth its characteristic of the Infinitesimal Calculus, and returns to the province of ordinary algebra.

The method of limits, or of prime and ultimate ratios, does not dispense with the introduction of these "assigned" and "unassigned" quantities: for the limit of a quantity is but the term to which this other quantity is supposed to approach continually, and it differs from it as little as we please. This limit is then considered as fixed and determinate or as an "assigned" quantity, while the other, being capable of approaching to this limit, as nearly as we please, remains always arbitrary or "unassigned," and subject only to the will of the calculator. Hence we may observe that the conception of limits is neither more nor less difficult to define exactly than that of an infinitely small quantity, and that consequently it is erroneous to think that the method of limits is more rigorous than that of the Infinitesimal Analysis. It is [illegible] [illegible] the method of limits [illegible] what is a limit. Now the difference between any quantity and its limit is exactly that which we should call an infinitely small quantity: the one is not therefore more difficult to understand than the other; and if the method of limits is exact, as we cannot doubt, there is no reason to doubt whether the Infinitesimal Analysis is so. But the latter has great advantages over the former; because in the method of limits we do not allow ourselves to introduce these semi-arbitrary quantities separately into the calculation; we do not admit even their ratios, but merely the limits of their ratios, which are "unassigned" quantities. Consequently we are deprived of those means of combination and transformation, which procure for the Infinitesimal Analysis the property (which it claims to itself, and that justly) of acting on these auxiliary quantities separately, a property which constitutes one of the principal advantages of its notation. The Infinitesimal Analysis is therefore an improvement on the method of limits; it is a more general application of the latter, and such as nevertheless is neither less exact nor less clear.

In conclusion, it is not in the explanation of the principles that one can shew the advantage of the Infinitesimal Analysis over all others. All of them are for the most part equally clear in their principles: but it is not equally easy to apply them to particular questions. The principal difficulty is to put problems into equations, which, on the other hand, in the Infinitesimal Analysis is very easy: the other methods refuse to admit into their Calculus the species of quantities, which we have called infinitely small, and their means of comparison are limited: they are obliged to go round in order to arrive at the same end; and this difficulty is much less observable in algebraic phrases and in operations of any kind whatever, than in propositions or arguments which precede them, or are substituted for these operations. One example will be sufficient to prove the superiority of the Infinitesimal Analysis in this respect. Let us take the enunciation of the famous principle of virtual velocities. The following is given by Lagrange in his "Mecanique Analytique."

"If any system whatever of as many bodies or points as we please, acted on by any forces whatever, be in equilibrio, and if we give to this system any small motion whatsoever, in consequence of which each point describes an infinitely small space, which will express its virtual velocity; the sum of the forces, multiplied each by the space which the point to which it is applied, describes in the direction of this same force, will equal zero: taking as positive the small spaces described in the path of the forces, and as negative the spaces described in an opposite direction."

Now how will those who reject the expressions admitted into the Infinitesimal Analysis enunciate this proposition, as clearly as we have just done after the example of the celebrated author of the Mecanique Analytique? But all Mathematics are, properly speaking, merely a course of similar expressions: to abandon them would be to plunge into tedious and intricate difficulties: we should still have to dread least some errors might remain in the result: now every one allows the method to be infallible in its results. It must be observed, that in mathematical researches we naturally fix our minds on quantities called infinitely small, and not on the limits of their ratios. If the volume of a body having a curved surface be required, we really imagine this body divided into a great number of slices, or even of particles: these slices or these particles are what we in fact consider, and not the different ratios which they may have to one another, and still

less the limits of these ratios: the imagination looks for an object of sense: purely algebraic forms offer to it merely what is vague. The division of volume into layers or particles presents a picture to the mind, clears and directs its operations, and facilitates the solution. One of these particles is regarded as the element of the whole quantity, which is in fact supposed to be the sum of all these elements; the differential expression is then sought out, which is to represent this particle, neglecting that which the rules of the Calculus justify omitting. The known formulas of the integral Calculus are in the next place applied to this differential expression, and thus we are enabled without much trouble to solve a problem, which perhaps would have resisted every effort of the method of exhaustion, or of any other which does not allow of using those means of abbreviation and simplification which are supplied by the Infinitesimal Analysis. It is allowable to consider infinitely small quantities either as real quantities, or as absolutely nothing: but the imagination accommodates itself more readily to the method which looks upon objects as efficient, than to that which supposes them reduced to zero. The law of continuity itself,

which alone can determine the value of the fractions $\frac{o}{o}$ in each case, whereas without it they would remain indeterminate, oblige us to compare them before they vanish altogether: besides, all these quantities may be eliminated, and that without assigning to them any determinate value; it is therefore at any rate superfluous to suppose them equal to zero: it is particularizing the question when we might dispense with doing so, and consequently it is to solve it in a less elegant way.

The essential, the sublime merit, we might almost say, of the Infinitesimal Analysis, is the union of the facility in common processes of approximation with the exactness of results in common analysis. This immense advantage would be lost, or at any rate diminished, if, instead of this pure and simple method such as Leibnitz supplied, we were to substitute for it other less natural means under the appearance of maintaining greater rigour throughout the course of the calculations; means which would be less convenient, less in conformity with the probable progress of the discoverers. If this method is exact in its results, as no one of the present day doubts, if we must always have recourse to it in difficult questions, as has likewise been allowed,

less the limits of these ratios: the imagination looks for an object of sense: purely algebraic forms offer to it merely what is vague. The division of volume into layers or particles presents a picture to the mind, clears and directs its operations, and facilitates the solution. One of these particles is regarded as the element of the whole quantity, which is in fact supposed to be the sum of all these elements; the differential expression is then sought out, which is to represent this particle, neglecting that which the rules of the Calculus justify omitting. The known formulas of the integral Calculus are in the next place applied to this differential expression, and thus we are enabled without much trouble to solve a problem, which perhaps would have resisted every effort of the method of exhaustion, or of any other which does not allow of using those means of abbreviation and simplification which are supplied by the Infinitesimal Analysis. It is allowable to consider infinitely small quantities either as real quantities, or as absolutely nothing: but the imagination accommodates itself more readily to the method which looks upon objects as efficient, than to that which supposes them reduced to zero. The law of continuity itself,

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It has been remarked many times by this profound thinker, that the real secret of analysis consists in the art of making one's self acquainted with the different degrees of indetermination, of which the quantity is capable; an idea which always struck us as forcible, and which occasioned our regarding the method of indeterminate quantities of Descartes as the most important corollary to the method of exhaustion.

In all branches of analysis taken generally, we observe that these operations are always founded on the different degrees of indetermination in the quantities compared by it. An abstract number is less determinate than a concrete, since the latter determines not only the quantity but the quality of the object submitted to the calculus. Algebraic quantities are more indeterminate than abstract numbers, because they do not specify the quantity. Amongst these last, variables are more indeterminate than constants, since the latter are considered fixed for a longer period in the calculation. Infinitesimal quantities are more indeterminate than simple variables, since they still continue susceptible of change, even when it has already been agreed on to consider the

others as fixed. Lastly, the variations are more indeterminate than the simple differentials, since the latter are restricted to varying according to a given law, instead of which the law according to which the others change is arbitrary. This gradation in the different degrees of indetermination is endless, and it is in this union of quantities more or less defined, more or less arbitrary, that is founded the fertile principle of the general method of indeterminate quantities, of which the Infinitesimal Calculus is in truth but a fortunate application.

These quantities, on one hand united to the conditions of the question, whilst on the other we are free to attribute to them greater or less values, these quantities, which are in some sort arbitrary, make us observe the necessity of drawing that distinction which we have made between "assigned" and "unassigned" quantities; a distinction which is not the same as that between constants and variables; for "assigned" quantities comprise under their term the unknown constants and variables, and their functions; that is to say, all those which can enter into the results of the Calculus, whilst "unassigned" quantities are necessarily excluded from them. The latter can enter then merely as auxiliaries, serving

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In all branches of analysis taken generally, we observe that these operations are always founded on the different degrees of indetermination in the quantities compared by it. An abstract number is less determinate than a concrete, since the latter determines not only the quantity but the quality of the object submitted to the calculus. Algebraic quantities are more indeterminate than abstract numbers, because they do not specify the quantity. Amongst these last, variables are more indeterminate than constants, since the latter are considered fixed for a longer period in the calculation. Infinitesimal quantities are more indeterminate than simple variables, since they still continue susceptible of change, even when it has already been agreed on to consider the

others as fixed. Lastly, the variations are more indeterminate than the simple differentials, since the latter are restricted to varying according to a given law, instead of which the law according to which the others change is arbitrary. This gradation in the different degrees of indetermination is endless, and it is in this union of quantities more or less defined, more or less arbitrary, that is founded the fertile principle of the general method of indeterminate quantities, of which the Infinitesimal Calculus is in truth but a fortunate application.

These quantities, on one hand united to the conditions of the question, whilst on the other we are free to attribute to them greater or less values, these quantities, which are in some sort arbitrary, make us observe the necessity of drawing that distinction which we have made between "assigned" and "unassigned" quantities; a distinction which is not the same as that between constants and variables; for "assigned" quantities comprise under their term the unknown constants and variables, and their functions; that is to say, all those which can enter into the results of the Calculus, whilst "unassigned" quantities are necessarily excluded from them. The latter can enter then merely as auxiliaries, serving

only to facilitate the expression of the conditions of the problem: after which all the energies of the calculator must be employed with a view to their elimination, which is indispensable in every case, and always shews, when it is completed, that this calculation loses from that time forth its characteristic of the Infinitesimal Calculus, and returns to the province of ordinary algebra.

The method of limits, or of prime and ultimate ratios, does not dispense with the distinction of these "assigned" and "unassigned" quantities; for the limit of a quantity is but the term to which this other quantity is supposed to approach continually, until it differs from it as little as we please. This limit is then considered as fixed, and consequently as an "assigned" quantity, whilst the other, being capable of approaching to this limit as much as we please, remains always arbitrary or "unassigned," and cannot enter into the result of the calculation. Hence we may observe, that the expression of limit is neither more nor less difficult to define exactly than that of an infinitely small quantity; and that consequently it is erroneous to think that the method of limits is more rigorous than that of the Infinitesimal Analysis: for, in order to proceed strictly by the method of limits, it is necessary first to define what is a limit. Now the difference between any quantity and its limit is exactly that which we should call an infinitely small quantity: the one is not therefore more difficult to understand than the other; and if the method of limits is exact, as we cannot doubt, there is no reason to doubt whether the Infinitesimal Analysis is so. But the latter has great advantages over the former; because in the method of limits we do not allow ourselves to introduce these semi-arbitrary quantities separately into the calculation; we do not admit even their ratios, but merely the limits of their ratios, which are "unassigned" quantities. Consequently we are deprived of those means of combination and transformation, which procure for the Infinitesimal Analysis the property (which it claims to itself, and that justly) of acting on these auxiliary quantities separately, a property which constitutes one of the principal advantages of its notation. The Infinitesimal Analysis is therefore an improvement on the method of limits; it is a more general application of the latter, and such as nevertheless is neither less exact nor less clear.

In conclusion, it is not in the explanation of the principles that one can shew the advantage of the Infinitesimal Analysis over all others. All of them are for the most part equally clear in their principles: but it is not equally easy to apply them to particular questions. The principal difficulty is to put problems into equations, which, on the other hand, in the Infinitesimal Analysis is very easy: the other methods refuse to admit into their Calculus the species of quantities, which we have called infinitely small, and their means of comparison are limited: they are obliged to go round in order to arrive at the same end; and this difficulty is much less observable in algebraic phrases and in operations of any kind whatever, than in propositions or arguments which precede them, or are substituted for these operations. One example will be sufficient to prove the superiority of the Infinitesimal Analysis in this respect. Let us take the enunciation of the famous principle of virtual velocities. The following is given by Lagrange in his "Mecanique Analytique."

"If any system whatever of as many bodies or points as we please, acted on by any forces whatever, be in equilibrio, and if we give to this system any small motion whatsoever, in consequence of which each point describes an infinitely small space, which will express its virtual velocity; the sum of the forces, multiplied each by the space which the point to which it is applied, describes in the direction of this same force, will equal zero: taking as positive the small spaces described in the path of the forces, and as negative the spaces described in an opposite direction."

Now how will those who reject the expressions admitted into the Infinitesimal Analysis enunciate this proposition, as clearly as we have just done after the example of the celebrated author of the Mecanique Analytique? But all Mathematics are, properly speaking, merely a course of similar expressions: to abandon them would be to plunge into tedious and intricate difficulties: we should still have to dread least some errors might remain in the result: now every one allows the method to be infallible in its results. It must be observed, that in mathematical researches we naturally fix our minds on quantities called infinitely small, and not on the limits of their ratios. If the volume of a body having a curved surface be required, we really imagine this body divided into a great number of slices, or even of particles: these slices or these particles are what we in fact consider, and not the different ratios which they may have to one another, and still

less the limits of these ratios: the imagination looks for an object of sense: purely algebraic forms offer to it merely what is vague. The division of volume into layers or particles presents a picture to the mind, clears and directs its operations, and facilitates the solution. One of these particles is regarded as the element of the whole quantity, which is in fact supposed to be the sum of all these elements; the differential expression is then sought out, which is to represent this particle, neglecting that which the rules of the Calculus justify omitting. The known formulas of the integral Calculus are in the next place applied to this differential expression, and thus we are enabled without much trouble to solve a problem, which perhaps would have resisted every effort of the method of exhaustion, or of any other which does not allow of using those means of abbreviation and simplification which are supplied by the Infinitesimal Analysis. It is allowable to consider infinitely small quantities either as real quantities, or as absolutely nothing: but the imagination accommodates itself more readily to the method which looks upon objects as efficient, than to that which supposes them reduced to zero. The law of continuity itself,

which alone can determine the value of the fractions $\frac{o}{o}$ in each case, whereas without it they would remain indeterminate, oblige us to compare them before they vanish altogether: besides, all these quantities may be eliminated, and that without assigning to them any determinate value; it is therefore at any rate superfluous to suppose them equal to zero: it is particularizing the question when we might dispense with doing so, and consequently it is to solve it in a less elegant way.

The essential, the sublime merit, we might almost say, of the Infinitesimal Analysis, is the union of the facility in common processes of approximation with the exactness of results in common analysis. This immense advantage would be lost, or at any rate diminished, if, instead of this pure and simple method such as Leibnitz supplied, we were to substitute for it other less natural means under the appearance of maintaining greater rigour throughout the course of the calculations; means which would be less convenient, less in conformity with the probable progress of the discoverers. If this method is exact in its results, as no one of the present day doubts, if we must always have recourse to it in difficult questions, as has likewise been allowed, why should we return to indirect and complicated methods to supply its place? Why be contented with relying on its deductions and the conformity of its results with those which the other methods furnish, when we can demonstrate it directly and generally, perhaps more readily, than by any of these methods? The objections brought against it all go on this false supposition, that the errors made in the course of the calculation by neglecting infinitely small quantities, remain in the result of the Calculus, however small we may suppose them. Now this is not the case; elimination removes them entirely; and it is singular, that in this indispensable condition of elimination the real character of Infinitesimal Quantities, and a decisive answer to all objections, should not hitherto have been discovered.

THE END.

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Fig. 1.

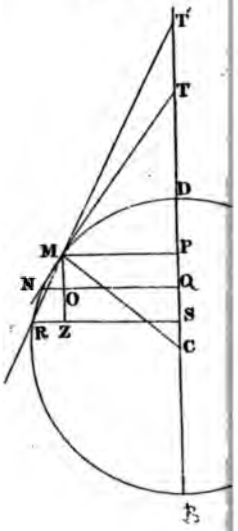
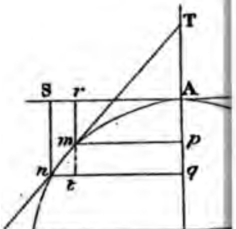


Fig. 3.



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